Lec 2 Risk Aversion

§1

Def A decision maker is risk averse if :

$\forall F(\cdot)$ the degenerate lottery with $\int x \, dF(x)$ for certain is at least as good as $F(\cdot)$.

If preferences admit an expected utility representation with Bernoulli utility function $u(\cdot)$ then the decision maker is risk averse iff

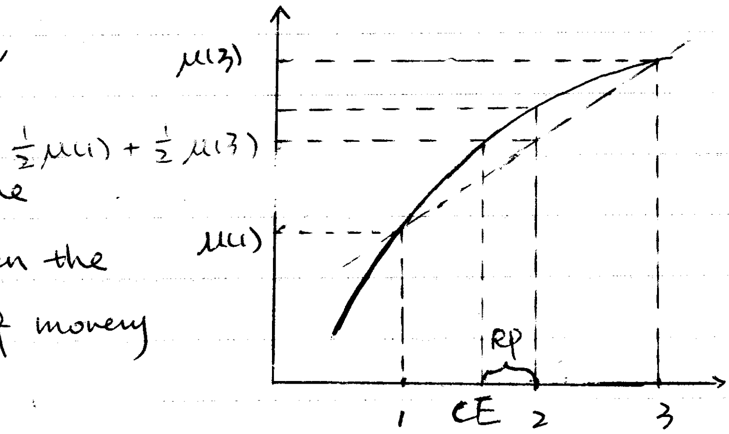
$$\int u(x) \, dF(x) \leq u\left(\int x \, dF(x) \right) \quad \text{(def of concave function)}$$

$\Leftrightarrow$ Bernoulli utility is concave, which is $u''(\cdot) \leq 0$.

Def A decision maker is: 1> risk neutral if she is always indifferent between $F(\cdot)$ and degenerate lottery with $\int x \, dF(x)$ for certain.

2> strictly risk averse if the decision maker is risk averse and indifference holds only when two lotteries are the same.

Def The certainty equivalence of $F(\cdot)$, denoted by $CE(F, u)$ is the amount of money to which the individual is indifferent between the lottery $F(\cdot)$ and the amount of money $CE(F, u)$, that is



$$u(CE(u, F)) = \int u(x) \, dF(x).$$

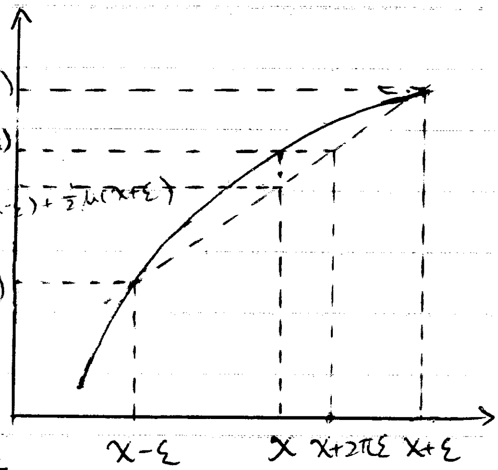
Example: lottery: $x = \begin{cases} 1 & \text{with } p = \frac{1}{2} \\ 3 & \text{with } p = \frac{1}{2} \end{cases}$

Risk premium: $RP(F, u) = \int x \, dF(x) - CE(F, u)$

is the maximum amount that the decision maker is willing to pay in order to avoid the risk associated with $F(\cdot)$.

intuition: if the payoff is given for sure, even if it is less than the expected payoff of the lottery, the decision make would accept it without playing as long as it is larger than CE

Def For fixed x, $\varepsilon > 0$. the probability premium, denoted by $\pi(x, \varepsilon, \mu)$, is the excess of winning probability over fair odds that makes the decision maker indifferent between the certain outcome x and a gamble between 2 outcomes. $x+\varepsilon$, $x-\varepsilon$.



i.e. $\mu(x) = \left[\frac{1}{2} + \pi(x, \varepsilon, \mu)\right] \mu(x+\varepsilon) + \left[\frac{1}{2} - \pi(x, \varepsilon, \mu)\right] \mu(x-\varepsilon)$

Prop: Suppose a decision maker is an expected utility maximizer with Bernoulli utility function $\mu(\cdot)$. then the followings are equivalent:

Equivalence characterization of risk aversion

1°: the decision maker is risk averse.

2°: $\mu(\cdot)$ is concave

3°: $CE(F, \mu) \le \int x \, dF(x)$, $\forall F(\cdot)$

4°: $RP(F, \mu) \geq 0$ $\forall F(\cdot)$

5°: $\pi(x, \varepsilon, \mu) \geq 0$ $\forall x, \varepsilon > 0$.

proof: Only part to be proven is: $\pi(x, \varepsilon, \mu) \geq 0$. $\forall x, \varepsilon$.

$\Leftrightarrow$ μ is concave.

($\Leftarrow$)

$$\mu(x) = \underbrace{\left[\tfrac{1}{2} + \pi(x, \varepsilon, \mu)\right]}_{\alpha} \mu(x+\varepsilon) + \underbrace{\left[\tfrac{1}{2} - \pi(x, \varepsilon, \mu)\right]}_{1-\alpha} \mu(x-\varepsilon)$$

Concavity of μ

$$\le \mu\left[\left(\tfrac{1}{2} - \pi(x, \varepsilon, \mu)\right)(x-\varepsilon) + \left[\tfrac{1}{2} + \pi(x, \varepsilon, \mu)\right](x+\varepsilon)\right]$$

$$= \mu(x + 2\pi\varepsilon).$$

Since $\mu(\cdot)$ is an increasing function, then $2\pi\varepsilon \geq 0$ $\triangle$

i.e. $\pi \geq 0$.

$x_1 \geq x_2$

($\Rightarrow$) Suppose $\pi(x, \varepsilon, \mu) \geq 0$ $\forall x, \varepsilon$. For any $x_1, x_2 \in \mathbb{R}^+$ WLOG.

$$\mu(\tfrac{1}{2}x_1 + \tfrac{1}{2}x_2) = \left[\tfrac{1}{2} - \pi(\tfrac{x_1+x_2}{2}, \varepsilon, x)\right]\mu(\tfrac{x_1+x_2}{2} - \varepsilon) + \left[\tfrac{1}{2} + \pi(\tfrac{x_1+x_2}{2}, \varepsilon, \mu)\right]\mu(\tfrac{x_1+x_2}{2} + \varepsilon)$$

by letting $\varepsilon = \frac{x_1 - x_2}{2}$

$$= (\tfrac{1}{2} - \pi)\mu(x_2) + (\tfrac{1}{2} + \pi)\mu(x_1)$$

$$= \tfrac{1}{2}(\mu(x_2) + \mu(x_1)) + (\mu(x_1) - \mu(x_2))\pi$$

$$\geq \tfrac{1}{2}\mu(x_1) + \tfrac{1}{2}\mu(x_2).$$

Since $\mu(\cdot)$ is continuous, $\mu(\cdot)$ is concave. ◇

Recall the Von Neumann-Morgenstern expected utility function is linear and hence cts. WLOG. assume Bernoulli utility function is cts.

§2. Comparison across individual

Motivation: Among different decision makers we may say one is "more" risk averse than the other. It is thus important to characterize the degree of risk aversion and make comparison.

Def Given Bernoulli utility function $\mu \in \mathcal{C}^2$. the Arrow-Pratt coeff of absolute risk aversion at x is

$$\gamma_A(x) = -\frac{\mu''(x)}{\mu'(x)}$$

(the rate at which the probability premium increases at certainty with small risk ε)

Derivation of γ_A: twice differentiate ⊛ w.r.t ε

⊛ $\mu(x) = [\tfrac{1}{2} - \pi(x, \varepsilon, \mu)]\mu(x-\varepsilon) + [\tfrac{1}{2} + \pi(x, \varepsilon, \mu)]\mu(x+\varepsilon)$,

$$\Rightarrow 0 = -\pi''(x,\varepsilon,\mu)\mu(x-\varepsilon) + 2\pi'(x,\varepsilon,\mu)\mu'(x-\varepsilon) + \mu''(x-\varepsilon)(\tfrac{1}{2} - \pi)$$

$$+ \pi''(x,\varepsilon,\mu)\mu(x+\varepsilon) + 2\pi'(x,\varepsilon,\mu)\mu'(x+\varepsilon) + \mu''(x+\varepsilon)(\tfrac{1}{2} + \pi)$$

Assume $\pi''(x, \varepsilon; \mu)$ is continuous, letting $\varepsilon \to 0$

$$0 = 4\pi'(x, 0, \mu)\mu'(x) + \mu''(x) \Rightarrow 4\pi'(x, 0, \mu) = -\frac{\mu''(x)}{\mu'(x)} = \gamma_A$$

[equivalence charaterization of u_2 is "more" risk averse than u_1]

Thm : The followings are equivalent :

1°: $\gamma_A(x, u_2) \geq \gamma_A(x, u_1) \quad \forall x \in \mathbb{R}^+$

2°: $\exists$ an increasing concave function $\varphi(\cdot)$ s.t

$$u_2(x) = \varphi(u_1(x)), \ \forall x.$$

i.e. u_2 is a concave transformation of u_1

("more" concave than u_1)

3° $CE(F, u_2) \leq CE(F, u_1) \quad \forall F(\cdot)$

4° $\pi(x, \varepsilon; u_2) \geq \pi(x, \varepsilon, u_1) \quad \forall x, \varepsilon.$

Proof : $(1^\circ \Rightarrow 2^\circ)$ $\exists \underset{\text{increasing}}{\varphi(\cdot)}$ s.t $u_2(x) = \varphi[u_1(x)]$ since both u_1, u_2 increasing ✓

twice differentiating $u_2(x) = \varphi[u_1(x)]$ then

$$u_2''(x) = \varphi''[u_1(x)][u_1'(x)]^2 + \varphi'[u_1(x)]u_1''(x)$$

Divide both sides by $u_2'(x)$ and note $u_2'(x) = \varphi'(u_1(x))\, u_1'(x)$,

$$\gamma_A(x, u_2) - \gamma_A(x, u_1) = -\frac{\varphi''[u_1(x)]}{\varphi'[u_1(x)]} \cdot u_1'(x) \geq 0$$

i.e. $\varphi''[u_1(x)] \leq 0$, φ is concave $_\Delta$

$(2^\circ \Rightarrow 3^\circ)$ $\varphi[u_1(CE(F, u_2))] = \int u_2(x)\, dF(x)$

$$= \int \varphi[u_1(x)]\, dF(x) \leq \varphi\left(\int u_1(x)\, dF(x)\right) = \varphi[u_1(CE(F, u_1))]$$

make use of concavity of φ

Since both φ and u_1 increasing, then

$u_1[CE(F, u_2)] \leq u_1[CE(F, u_1)]$, $CE(F, u_2) \leq CE(F, u_1)$ $_\Delta$

$(3° \Rightarrow 4°)$ Consider a lottery F:

payoff $= x+\varepsilon$ with probability $\frac{1}{2}+\pi(x,\varepsilon,u_1)$

$= x-\varepsilon$ with probability $\frac{1}{2}-\pi(x,\varepsilon,u_1)$

by definition, $u_1(x) = [\frac{1}{2}-\pi(x,\varepsilon,u_1)]u_1(x-\varepsilon) + [\frac{1}{2}+\pi(x,\varepsilon,u_1)]u_1(x+\varepsilon)$

i.e. $CE(F,u_1) = x$.

$$u_2[CE(F,u_2)] = [\tfrac{1}{2}-\pi(x,\varepsilon,u_1)]u_2(x-\varepsilon) + [\tfrac{1}{2}+\pi(x,\varepsilon,u_1)]u_2(x+\varepsilon)$$

$$\leq u_2[CE(F,u_1)] = u_2(x)$$

because u_2 increasing

and $CE(F,u_2) \leq CE(F,u_1)$

$$= [\tfrac{1}{2}-\pi(x,\varepsilon,u_2)]u_2(x-\varepsilon) + [\tfrac{1}{2}+\pi(x,\varepsilon,u_2)]u_2(x+\varepsilon)$$

then $\frac{1}{2}[u_2(x-\varepsilon)+u_2(x+\varepsilon)] + \pi(x,\varepsilon,u_1)(u_2(x+\varepsilon)-u_2(x-\varepsilon))$

$$\leq \tfrac{1}{2}[u_2(x-\varepsilon)+u_2(x+\varepsilon)] + \pi(x,\varepsilon,u_2)(u_2(x+\varepsilon)-u_2(x-\varepsilon))$$

$\Rightarrow$ $\pi(x,\varepsilon,u_1) \leq \pi(x,\varepsilon,u_2)$ since $u_2(x+\varepsilon) \geq u_2(x-\varepsilon)$ Δ

$(4° \Rightarrow 1°)$

$$\gamma_A(x,u_1) = 4\pi'(x,0,u)$$

$$= 4\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}(\pi(x,\varepsilon,u_1) - \overbrace{\pi(x,0,u_1)}^{=0})$$

$$= 4\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\pi(x,\varepsilon,u_1) \leq 4\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\pi(x,\varepsilon,u_2)$$

$$= 4\pi'(x,0,u_2) = \gamma_A(x;u_2)\ \Delta$$

§ Comparison across wealth levels.

Def The Bernoulli utility function $u(\cdot)$ exhibits: 1) decreasing absolute risk aversion if $\gamma_A(x)$ is a decreasing function of x.

2) Constant absolute risk aversion if $\gamma_A(x)$ is constant over x.

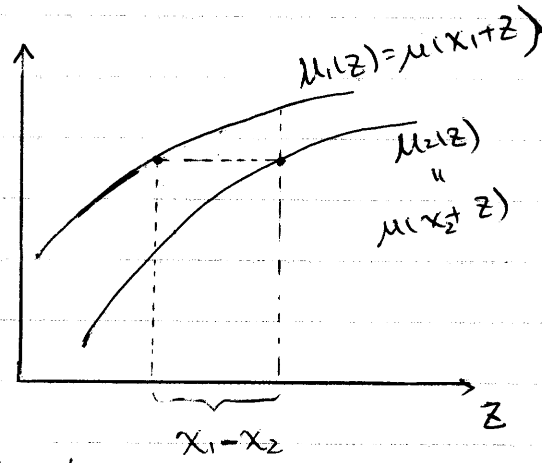
3) increasing absolute risk aversion if $\gamma_A(x)$ is increasing in x.

Motivation: People may behave in different manners when they have different amount of wealth. Intuitively, wealthier people may willing to take more risks because they can afford the outcome.

Consider two initial wealth level $x_1 > x_2$ induced Bernoulli utility functions

$$u_1(z) = u(x_1 + z), \quad u_2(z) = u(x_2 + z)$$

Comparing $u_1(\cdot), u_2(\cdot)$ is essentially comparing an individual's attitutes towards risk as his level of wealth changes.



Thm the following statements are equivalent:

1° The Bernoulli utility function $u(\cdot)$ exhibits decreasing absolute risk aversion.

2°. $r_A(z, x_2) = -\dfrac{u''(x_2+z)}{u'(x_2+z)} \geq r_A(z, x_1) = -\dfrac{u''(x_1+z)}{u'(x_1+z)} \quad \forall x_1 \geq x_2$

3°. $\exists$ an increasing concave function $\varphi(\cdot)$ s.t

$$u_2(z) = \varphi(u_1(z)) \quad \forall z.$$

4° $CE(F, u_2) \leq CE(F, u_1) \quad \forall F(\cdot)$ where

$CE(F, u_i) = CE(F, x_i)$ s.t $u(x_i + CE(F, x_i)) = \int u(x_i + t)\, dF(t)$

5° $\pi(x, \varepsilon, u_2) \geq \pi(x, \varepsilon, u_1) \quad \forall x, \varepsilon$ where

$\pi(x, \varepsilon, u_i) = \pi(x + x_i, \varepsilon, u)$, i.e the probability premium $\pi(x, \varepsilon, u)$ is decreasing in x.

Proof: $1° \Leftrightarrow 2°$ by definition. $5° \Rightarrow 2°$ is trivial, directly from definition,

$(2^\circ \Rightarrow 3^\circ)$ $\exists$ increasing function φ s.t. $u(x_2+z) = \varphi[u(x_1+z)]$

Since $u(x_1+z)$, $u(x_2, z)$ are ordinally identical. By differentiation wrt z

$$u'(x_2+z) = \varphi'[u(x_1+z)]\, u'(x_1+z) \qquad (1)$$

$$u''(x_2+z) = \varphi''[u(x_1+z)][u'(x_1+z)]^2 + \varphi'[u(x_1+z)]\, u''(x_1+z) \qquad (2)$$

divide both sides of (2) by $u'(x_2+z)$, make use of (1).

$$\gamma_A(z, x_2) = \gamma_A(z, x_1) - \frac{\varphi''[u_1(z)]}{\varphi'[u_1(z)]}\, u_1'(z)$$

Since $\gamma_A(z, x_2) \geq \gamma_A(z, x_1)$, φ increasing, u_1 increasing, then

$$\varphi''[u_1(z)] \leq 0, \quad \varphi \text{ is concave } \triangle$$

$(3^\circ \Rightarrow 4^\circ)$ $\forall F(\cdot)$, $u(x_2 + CE(F, x_2)) = \int u(x_2+t)\, dF(t)$.

$$u(x_1 + CE(F, x_1)) = \int u(x_1+t)\, dF(t).$$

$$\varphi[u(x_1 + CE(F, x_2))] = u(x_2 + CE(F, x_2)) = \int u(x_2+t)\, dF(t)$$

$$= \int \varphi[u(x_1+t)]\, dF(t)$$

$$\leq \varphi\left[\int u(x_1+t)\, dF(t)\right] = \varphi[u(x_1 + CE(F, x_1))]$$

then $u(x_1 + CE(F, x_2)) \leq u(x_1 + CE(F, x_1))$ by φ is increasing

$CE(F, x_2) \leq CE(F, x_1)$ by u is increasing $\triangle$

$(4^\circ \Rightarrow 5^\circ)$ Consider a lottery F with

payoff $= z + \varepsilon$ with probability $\frac{1}{2} + \pi(z, \varepsilon, x_1)$.

$= z - \varepsilon$ with probability $\frac{1}{2} - \pi(z, \varepsilon, x_1)$

Note $u(x_1+z) = [\frac{1}{2} + \pi(z, \varepsilon, x)]\, u(x_1+z+\varepsilon) + [\frac{1}{2} - \pi(z, \varepsilon, x_1)]\, u(x_1+z-\varepsilon)$

where $z = CE(F, x_1)$

$$u(x_2 + CE(F, x_1)) = \left[\tfrac{1}{2} - \pi(z, \varepsilon, x_1)\right] u(x_2 + z - \varepsilon) + \left[\tfrac{1}{2} + \pi(z, \varepsilon, x_1)\right] u(x_2 + z + \varepsilon)$$

$$= \tfrac{1}{2}\left[u(x_2+z-\varepsilon) + u(x_2+z+\varepsilon)\right] + \pi(z, \varepsilon, x_1)\left[u(x_2+z+\varepsilon) - u(x_2+z-\varepsilon)\right]$$

$$\leq u(x_2 + CE(F, x_1)) = u(x_2 + z)$$

$$= \left[\tfrac{1}{2} - \pi(z, \varepsilon, x_2)\right] u(x_2+z-\varepsilon) + \left[\tfrac{1}{2} + \pi(z, \varepsilon, x_2)\right] u(x_2+z+\varepsilon)$$

$$= \tfrac{1}{2}\left[u(x_2+z-\varepsilon) + u(x_2+z+\varepsilon)\right] + \pi(z, \varepsilon, x_2)\left[u(x_2+z+\varepsilon) - u(x_2+z-\varepsilon)\right]$$

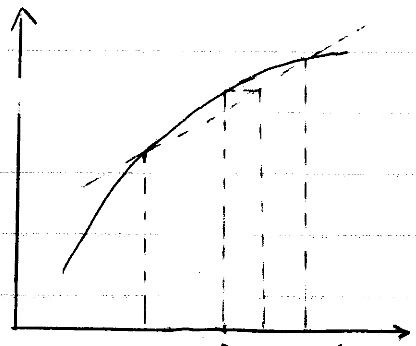
Since $u(x_2+z+\varepsilon) - u(x_2+z-\varepsilon) \geq 0$, then $\pi(z, \varepsilon, x_1) \leq \pi(z, \varepsilon, x_2)$

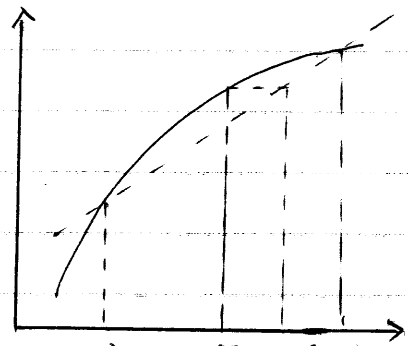
§4 $\diamond$

Def Given Bernoulli utility function $u(\cdot)$, the coefficient of relative risk aversion at x is

$$\gamma_R(x) = -x\frac{u''(x)}{u'(x)}$$

Deriaation:





risky projects whose outcome is absolute gains or losses from certain wealth.

risky projects whose outcome is ~~pertag~~ percentage gains or losses of current wealth.

$$u(x) = \left[\tfrac{1}{2} - \pi(x, \varepsilon x, u)\right] u(x - \varepsilon x) + \left[\tfrac{1}{2} + \pi(x, \varepsilon x, u)\right] u(x + \varepsilon x)$$

differentiating w.r.t ε twice, and letting $\varepsilon \to 0$

$$0 = 4\pi'(x, 0, u)\, u'(x)\, x + u''(x)\, x^2$$

$$\Rightarrow 4\pi'(x, 0, u) = -\frac{u''(x)}{u'(x)} \cdot x = \gamma_A(x) \cdot x \triangleq \gamma_R(x)$$

Remark: 1°. Comparison between individual

$$\gamma_R(x, \mu_2) \geq \gamma_R(x, \mu_1) \quad \forall x \geq 0 \iff \gamma_A(x, \mu_2) \geq \gamma_A(x, \mu_1)$$

2° Comparison between wealth

$$\text{DRRA}: \frac{\partial \gamma_R(x, \mu)}{\partial x} < 0 \Rightarrow \text{DARA}: \frac{\partial \gamma_A(x, \mu)}{\partial x} < 0$$

$$\not\Leftarrow$$

Lec 3 Portfolio Choice

§1 Model:
- decision maker has initial wealth w_0.
- there are two assets: risky asset and risk-free asset.
- r_f: the riskless interest rat
  $\tilde{r}$: the random rate of return on the risky asset

Suppose the individual invests S dollars in the risky asset

uncertain end-of-period wealth $\tilde{w} = (w_0 - S)(1+r_f) + S(1+\tilde{r})$

$= w_0(1+r_f) + S(\tilde{r} - r_f)$

Individual's choice problem

$$\max_{0 \le S \le w_0} E\left[\mu(w_0(1+r_f) + S(\tilde{r} - r_f))\right]$$

$$\parallel$$

$$\int \mu(w_0(1+r_f) + S(\tilde{r} - r_f))\, dF_{\tilde{r}}(\cdot)$$

F.O.C. $E[\mu'(\tilde{w})(\tilde{r} - r_f)] = 0$

S.O.C $E[\mu''(\tilde{w})(\tilde{r} - r_f)^2] < 0$ which is satisfied by assuming Bernoulli utility function is concave. i.e. $\mu'' < 0$

§2 findings from the model

① Prop (Participation) An individual who is risk averse, i.e. $\mu''(\cdot) < 0$, and strictly prefers more to less, i.e. $\mu'(\cdot) > 0$, will undertake risky investment if and only if $E(\tilde{r}) > r_f$ (under what condition an investor is willing to participate)

Proof Suppose nothing invested in risky asset, i.e. $S = 0$

$$E[\mu'(\tilde{w})(\tilde{r} - r_f)] = \mu'(w_0(1+r_f))[E(\tilde{r}) - r_f].$$

If $E(\tilde{r}) > r_f$. then $E[\mu'(\tilde{w})(\tilde{r} - r_f)] > 0$ and note $E[\mu''(\tilde{w})(\tilde{r} - r_f)^2] < 0$

$\therefore S^* > 0$.

If $S^* > 0$. then

$$0 \le E\left[\mu'(w_0(1+r_f) + S^*(\tilde r - r_f))(\tilde r - r_f)\right] < E\left[\mu'(w_0(1+r_f))(\tilde r - r_f)\right] \quad \circledast$$

Since $E\left[\mu'(w_0(1+r_f) + S(\tilde r - r_f))(\tilde r - r_f)\right]$ decreases in S strictly.

by $\circledast$ $\mu'(w_0(1+r_f))\left[E(\tilde r) - r_f\right] > 0 \Rightarrow E(\tilde r) > r_f$

Since $\mu'(\cdot) > 0$ ◇

② Prop:

minimum risk premium

(local behavior) Under what conditions an investor would invest all in the risky asset. Condition (☆).

For an investor to invest all in the risky asset, it requires

$$E\left[\mu'(w_0(1+\tilde r))(\tilde r - r_f)\right] \ge 0$$

First order Taylor expansion of $\mu'(w_0(1+\tilde r))$ around $w_0(1+r_f)$:

$$\mu'(w_0(1+\tilde r)) = \mu'(w_0(1+r_f)) + \mu''(w_0(1+r_f))\, w_0(\tilde r - r_f) + o\left(w_0^2(\tilde r - r_f)^2\right)$$

then $E\left[\mu'(w_0(1+\tilde r))(\tilde r - r_f)\right]$

$$= E\left[\mu'(w_0(1+r_f))(\tilde r - r_f) + \mu''(w_0(1+r_f))\, w_0 (\tilde r - r_f)^2 + o\left((\tilde r - r_f)^3\right)\right]$$

$$= \mu'(w_0(1+r_f))\, E(\tilde r - r_f) + \mu''(w_0(1+r_f))\, w_0\, E\left((\tilde r - r_f)^2\right) + o\left(E((\tilde r - r_f)^3)\right)$$

≥ 0

Here we ignore the high order terms. (in general, it is not ignorable, unless it's quadratic or some other special structures)

$$\Rightarrow \mu'(w_0(1+r_f))\, E(\tilde r - r_f) + \mu''(w_0(1+r_f))\, w_0\, E\left((\tilde r - r_f)^2\right) \ge 0$$

$$\Rightarrow E(\tilde r - r_f) \ge -\frac{\mu''(w_0(1+r_f))}{\mu'(w_0(1+r_f))} \cdot w_0\, E\left((\tilde r - r_f)^2\right)$$

☆.

$$= \gamma_A(w_0(1+r_f))\, w_0\, E\left((\tilde r - r_f)^2\right) = \gamma_R(w_0(1+r_f))\, \frac{E\left[(\tilde r - r_f)^2\right]}{1+r_f} \quad ◇$$

③

Prop. (Comparison among individuals)

Suppose individual 2 is more risk averse than individual 1,

and $E(\tilde{r} - r_f) > 0$ (Both of them are willing to participate)

then individual 2 will invest less in the risky asset than 1.

proof: Consider $u_1(\cdot)$, $u_2(\cdot)$ satisfy $u_i'(\cdot) > 0$. $u_i''(\cdot) < 0$

and $u_2(\cdot) = \varphi[u_1(\cdot)]$ where φ is increasing, concave. i.e. $u_2(\cdot)$ is a concave transformation of $u_1(\cdot)$.

Suppose individual 1 invests S_1 in risky assets

~~than~~ F.O.C $E[u_1'(\underbrace{w_0(1+r_f) + S_1(\tilde{r} - r_f)}_{\triangleq \tilde{w}_1})(\tilde{r} - r_f)] = 0$ Since $E(\tilde{r} - r_f) > 0$

with $0 < S_1 \leq w_0$

Consider individual 2:

$$E[u_2'(w_0(1+r_f) + S_1(\tilde{r} - r_f))(\tilde{r} - r_f)]$$

$$= E[\varphi'(u_1(\tilde{w}_1))u_1'(\tilde{w}_1)(\tilde{r} - r_f)] = (*)$$

- Case I: $\tilde{r} - r_f > 0$, then

$$\tilde{w}_1 = w_0(1+r_f) + S_1(\tilde{r} - r_f) > w_0(1+r_f),$$

$$\Rightarrow \varphi'(u_1(\tilde{w}_1)) < \varphi'(u_1(w_0(1+r_f)))$$

$$\Rightarrow \varphi'(u_1(\tilde{w}_1))u_1'(\tilde{w}_1)(\tilde{r} - r_f) < \varphi'(u_1(w_0(1+r_f)))u_1'(\tilde{w}_1)(\tilde{r} - r_f)$$

- Case II: $\tilde{r} - r_f < 0$. then

$$\tilde{w}_1 = w_0(1+r_f) + S_1(\tilde{r} - r_f) < w_0(1+r_f)$$

$$\Rightarrow \varphi'(u_1(\tilde{w}_1)) > \varphi'(u_1(w_0(1+r_f)))$$

$$\Rightarrow \varphi'(u_1(\tilde{w}_1))u_1'(\tilde{w}_1)(\tilde{r} - r_f) < \varphi'(u_1(w_0(1+r_f)))u_1'(\tilde{w}_1)(\tilde{r} - r_f)$$

hence $(*) < \varphi'(u_1(w_0(1+r_f)))E(u_1'(\tilde{w}_1)(\tilde{r} - r_f)) \stackrel{\text{by F.O.C}}{=} 0$

$\Rightarrow S_2 < S_1$ ▫

④

Prop (Wealth effect) Under assumption $E(\tilde{r}-r_f)>0$

1° $ds^*/dw_0 > 0$ if DARA

i.e. decreasing absolute risk aversion $\Rightarrow$ the risky asset is a norm good.

2° $ds^*/dw_0 < 0$ if IARA

i.e. increasing absolute risk aversion $\Rightarrow$ the risky asset is a inferior good

3° $ds^*/dw_0 = 0$ if CARA

i.e. constant absolute risk aversion $\Rightarrow$ the demand for the risky asset is invariant w.r.t initial wealth.

Proof: Under assumption $E(\tilde{r}-r_f)=0$

F.O.C. $E\left(u'(w_0(1+r_f)+s^*(\tilde{r}-r_f))(\tilde{r}-r_f)\right)=0$

take total derivative:

$$E\Big(u''(w_0(1+r_f)+s^*(\tilde{r}-r_f))(\tilde{r}-r_f)(1+r_f)\Big)dw_0$$
$$+E\Big(u''(w_0(1+r_f)+s^*(\tilde{r}-r_f))(\tilde{r}-r_f)^2\Big)ds^*=0$$

$$\frac{ds^*}{dw_0}=E\left(u''(w^*)(\tilde{r}-r_f)(1+r_f)\right)\Big/\left[-E\left(u''(w^*)(\tilde{r}-r_f)^2\right)\right]$$

where $w^*=w_0(1+r_f)+s^*(\tilde{r}-r_f)$.

Note $u''(\cdot)<0$, then $-E\left(u''(w^*)(\tilde{r}-r_f)\right)>0$.

$$\mathrm{sign}\left(\frac{ds^*}{dw_0}\right)=\mathrm{sign}\left(E\left(u''(w^*)(\tilde{r}-r_f)\right)\right)$$
$$=\mathrm{sign}\left(E\left(-\gamma_A(w^*)\,u'(w^*)(\tilde{r}-r_f)\right)\right)$$

Suppose DARA:

Case I: $\tilde{r} - r_f > 0$. $w^* > w_0(1+r_f)$

$$\Rightarrow \gamma_A(w^*) < \gamma_A(w_0(1+r_f))$$

$$\Rightarrow -\gamma_A(w^*)u'(w^*)(\tilde{r}-r_f) > -\gamma_A(w_0(1+r_f))u'(w^*)(\tilde{r}-r_f)$$

Case II: $\tilde{r} - r_f < 0$. $w^* < w_0(1+r_f)$

$$\Rightarrow \gamma_A(w^*) > \gamma_A(w_0(1+r_f))$$

$$\Rightarrow -\gamma_A(w^*)u'(w^*)(\tilde{r}-r_f) > -\gamma_A(w_0(1+r_f))u'(w^*)(\tilde{r}-r_f)$$

$$\Rightarrow E(-\gamma_A(w^*)u'(w^*)(\tilde{r}-r_f)) > E(-\gamma_A(w_0(1+r_f))u'(w^*)(\tilde{r}-r_f))$$

$$= -\gamma_A(w_0(1+r_f))\underbrace{E(u'(w^*)(\tilde{r}-r_f))}_{=0 \text{ by F.O.C}} = 0$$

$\Rightarrow \dfrac{dS^*}{dw_0} > 0$ which proves 1°.

2° and 3° can be proven by same arguments. □

⑤ Prop (Wealth elasticity) Under assumption $E(\tilde{r}-r_f) > 0$

1° $\eta > 1$ if DRRA

㊟ i.e. Decreasing relative risk aversion ⇒ the proportion of the individual's initial wealth invested in risky assets will increase as $w_0 \uparrow$

2°. $\eta < 1$ if IRRA

i.e. IRRA ⇒ propotion ↘ as $w_0 \nearrow$

3°. $\eta = 1$ if CRRA i.e. propotion keep unchanged.

where wealth elasticity (of the demand for risky assets) is defined as: $\eta = \dfrac{dS^*/S^*}{dw_0/w_0} = \dfrac{dS^*}{dw_0} \cdot \dfrac{w_0}{S^*}$

Reasons for ⊛ : let $\theta^* = \frac{S^*}{w_0}$.

$$d\theta^*/dw_0 = \frac{1}{w_0^2}\left(ds^*/dw_0 \, w_0 - S^*\right) = \frac{S^*}{w_0^2}\left(\frac{ds^*}{dw_0}\frac{w_0}{S^*} - 1\right) = \frac{S^*}{w_0^2}(\eta - 1)$$

Since $E(\tilde{r}-r_f) > 0$. then $S^* > 0$.

$$\Rightarrow \begin{cases} d\theta^*/dw_0 > 0 & \text{if } \eta > 1 \quad (\text{DRRA}) \\ d\theta^*/dw_0 < 0 & \text{if } \eta < 1 \quad (\text{IRRA}) \\ d\theta^*/dw_0 = 0 & \text{if } \eta = 1 \quad (\text{CRRA}) \end{cases}$$

proof :
$$\eta = \frac{ds^*}{dw_0}\cdot\frac{w_0}{S^*} = 1 + \frac{1}{S^*}\left(\frac{ds^*}{dw_0}\cdot w_0 - S^*\right)$$

Note $\frac{ds^*}{dw_0} = E(u''(w^*)(1+r_f)(\tilde{r}-r_f))/[-E(u''(w^*)(\tilde{r}-r_f)^2)]$

as shown in previous proposition.

with $w^* = w_0(1+r_f) + S^*(\tilde{r}-r_f)$ and positive denominator since $u''(\cdot) < 0$

then
$$\eta = 1 + \frac{[w_0(1+r_f)E(u''(w^*)(\tilde{r}-r_f)) + S^* E(u''(w^*)(\tilde{r}-r_f)^2)]}{-S^* E(u''(w^*)(\tilde{r}-r_f)^2)}$$

$$= 1 + E(u''(w^*)w^*(\tilde{r}-r_f))/[-S^* E(u''(w^*)(\tilde{r}-r_f)^2)]$$

with $-S^* E(u''(w^*)(\tilde{r}-r_f)^2) > 0$ since $S^* > 0$.

then $\text{sign}(\eta - 1) = \text{sign}(E(u''(w^*)w^*(\tilde{r}-r_f)))$

$$= \text{sign}(E(-\gamma_R(w^*)\, u'(w^*)(\tilde{r}-r_f)))$$

Same as before. discuss two case $\tilde{r}-r_f > 0$ and $\tilde{r}-r_f < 0$.

we get $E(u''(w^*)w^*(\tilde{r}-r_f)) > -\gamma_R(w_0(1+r_f))\, E(u'(w^*)(\tilde{r}-r_f)) \overset{\text{F.O.C}}{=} 0$

when DRRA $\Rightarrow \eta > 1$.

⑥ Prop (Effect of the risk free interest rate) Given $0 \le S^* < w_0$

1°. $dS^*/dr_f < 0$ if IARA or IRRA or CARA or CRRA

2°. dS^*/dr_f has uncertain sign if DARA or DRRA

i.e. increase in the risk free interest rate has uncertain effect to the optimal investment in the risky asset

proof: Total differentiate F.O.C : $E(u'(w_0(1+r_f)+S^*(\tilde{r}-r_f))\cdot(\tilde{r}-r_f))=0$

$$E(u''(w_0(1+r_f)+S^*(\tilde{r}-r_f))(w_0-S^*)(\tilde{r}-r_f))\,dr_f - E(u'(w_0(1+r_f)+S^*(\tilde{r}-r_f)))dr_f$$

$$+ E(u''(w_0(1+r_f)+S^*(\tilde{r}-r_f))(\tilde{r}-r_f)^2)\,dS^* = 0$$

$$\Rightarrow \frac{dS^*}{dr_f} = \left[E(u''(w^*)(w_0-S^*)(\tilde{r}-r_f)) - E(u'(w^*))\right] \Big/ \left[-E(u''(w^*)(\tilde{r}-r_f)^2)\right]$$

Since $u''(\cdot)<0$. denominator $-E(u''(w^*)(\tilde{r}-r_f)^2)>0$

$$\Rightarrow \operatorname{sign}\left(\frac{dS^*}{dr_f}\right) = \operatorname{sign}\left((w_0-S^*)E(u''(w^*)(\tilde{r}-r_f)) - E(u'(w^*))\right) \quad ⊛$$

Note $E(u'(w^*))>0$. From proof in Prop ④.

$$\text{IARA} \Rightarrow E(u''(w^*)(\tilde{r}-r_f))<0 \Rightarrow \frac{dS^*}{dr_f}<0$$

$$\text{CARA} \Rightarrow E(u''(w^*)(\tilde{r}-r_f))=0 \Rightarrow \frac{dS^*}{dr_f}<0$$

$$\text{DARA} \Rightarrow E(u''(w^*)(\tilde{r}-r_f))>0 \Rightarrow \operatorname{sign}\left(\frac{dS^*}{dr_f}\right) \text{ uncertain.}$$

$$⊛ \Rightarrow \operatorname{sign}\left(\frac{dS^*}{dr_f}\right) = \operatorname{sign}\left(\frac{w_0-S^*}{w^*}E[u''(w^*)w^*(\tilde{r}-r_f)] - E(u'(w^*))\right)$$

From proof of Prop ⑤

$$\text{IRRA or CRRA} \Rightarrow dS^*/dr_f < 0$$

$$\text{DRRA} \Rightarrow \operatorname{sign}\left(\frac{dS^*}{dr_f}\right) \text{ uncertain.} \quad \square$$

① Prop (Effect from the expect return of the risk asset)

Under assumption $E(\tilde{r} - r_f) > 0$. then $S^* > 0$

let $\xi = E(\tilde{r})$. then:

$dS^*/d\xi > 0$ if DARA or CARA or DRRA or CRRA ; $\text{sign}(dS^*/d\xi)$ uncertain if IARA or IRRA

proof: Note $\tilde{r} - r_f = \xi + (\tilde{r} - \xi) - r_f = \xi + \varepsilon - r_f$

where $\varepsilon = \tilde{r} - \xi$ capturing all the uncertainty

F.O.C: $E\big(u'(w^*)(\xi + \varepsilon - r_f)\big) = 0$ where $w^* = w_0(1+r_f) + S^*(\xi + \varepsilon - r_f)$

total differentiation:

$$E\big(u''(w^*)S^*(\xi + \varepsilon - r_f)\big)d\xi + E(u'(w^*))d\xi + E\big(u''(w^*)(\tilde{r} - r_f)^2\big)dS^* = 0$$

$$\Rightarrow \frac{dS^*}{d\xi} = \frac{S^* E(u''(w^*)(\tilde{r} - r_f)) + E(u'(w^*))}{-E(u''(w^*)(\tilde{r} - r_f)^2)}$$

$$\Rightarrow \text{sign}\left(\frac{dS^*}{d\xi}\right) = \text{sign}\big(S^* E(u''(w^*)(\tilde{r} - r_f)) + E(u'(w^*))\big)$$

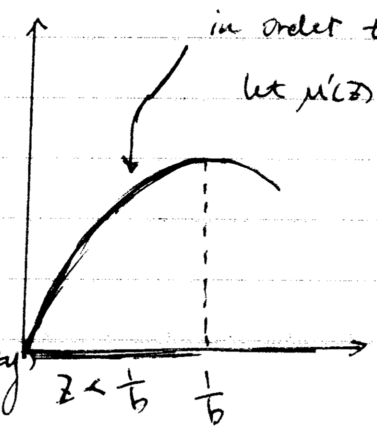
$$= \text{sign}\left(\frac{S^*}{w^*} E\big(u''(w^*)w^*(\tilde{r} - r_f)\big) + E(u'(w^*))\right)$$

$\Rightarrow$ if DARA or DRRA or CARA or CRRA $dS^*/d\xi > 0$

if IARA. $\text{sign}(dS^*/d\xi)$ is uncertain.

△ Commonly used utility functions

1° Concave quadratic utility function

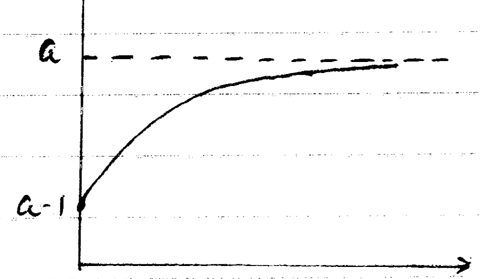


- $u(z) = z - \frac{1}{2}bz^2$. $u''(z) = -b < 0$ (concavity) $\Rightarrow b > 0$

$$\mu'(z) = be^{-bz} > 0$$

$$\mu''(z) = -b^2 e^{-bz} < 0$$

2°. Negative exponential utility function



$$\bullet\ \mu(z) = a - e^{-bz}, \quad b > 0$$

$$\gamma_A(z) = -\frac{\mu''(z)}{\mu'(z)} = b \quad \text{constant ARA}$$

$\rightarrow$ any change in initial wealth will be absorbed by riskless asset

$$\gamma_R(z) = \gamma_A(z) \cdot z = bz \quad \text{increasing RRA}$$

3° Narrow power utility function $\bullet\ \mu(z) = \frac{B}{B-1} z^{\frac{B-1}{B}}$ similar to CRRA utility

$$\mu'(z) = z^{-\frac{1}{B}} > 0, \quad \mu''(z) = -\frac{1}{B} z^{-\frac{1}{B}-1} < 0$$

$$\gamma_A(z) = \frac{1}{B} z^{-1} \quad \text{decreasing ARA}$$

$$\gamma_R(z) = \frac{1}{B}$$ constant RRA $\rightarrow$ propotion of wealth in the risky asset is invariant w.r.t changes in his initial wealth level.

4° Extended power utility function

$$\bullet\ \mu(z) = \frac{1}{B-1}(A+Bz)^{1-\frac{1}{B}}, \quad B > 0, \text{ and } z > \max\left[-\frac{A}{B}, 0\right]$$

$$\mu'(z) = (A+Bz)^{-\frac{1}{B}} > 0 \quad \text{ensured by } z > \max\left[-\frac{A}{B}, 0\right]$$

$$\mu''(z) = -(A+Bz)^{-\frac{1}{B}-1} < 0$$

$$\gamma_A(z) = (A+Bz)^{-1} \quad \text{decreasing ARA}$$

$$\gamma_R(z) = \frac{z}{A+Bz}, \quad \frac{d\gamma_R(z)}{dz} = \frac{A}{(A+Bz)^2} = \begin{cases} > 0 & \text{if } A > 0 \quad \text{IRRA} \\ = 0 & \text{if } A = 0 \quad \text{CRRA} \\ < 0 & \text{if } A < 0 \quad \text{DRRA} \end{cases}$$

§4 J-risky assets + Two-found separation

Model:

- initial wealth W_0 - r_f: the riskless interest rate
- $\tilde{r}_j$: the random rate of return on the j^{th} risky asset
- a_j: the dollar investment in j^{th} asset $\quad j = 1, 2, \cdots, J$

- The uncertain end-of-period wealth

$$\tilde{w} = \left(w_0 - \sum_{j=1}^{J} \alpha_j\right)(1+r_f) + \sum_{j=1}^{J} \alpha_j (1+\tilde{r}_j)$$

$$= w_0(1+r_f) + \sum_{j=1}^{J} \alpha_j (\tilde{r}_j - r_f)$$

- The individual's choice problem

$$\max_{\{\alpha_j\}} E\left[u\left(w_0(1+r_f) + \sum_{j=1}^{J} \alpha_j (\tilde{r}_j - r_f)\right)\right]$$

F.O.C. $E\left[u'(\tilde{w})(\tilde{r}_j - r_f)\right] = 0. \quad j = 1, 2, \dots, J$ Since u is Concave the FOC is also sufficient.

where $\tilde{w} = w_0(1+r_f) + \sum_{j=1}^{J} \alpha_j (\tilde{r}_j - r_f)$

Participation Condition conditions to make individuals have no intention to ivest in risky asset

$$E\left[u'(w_0(1+r_f))(\tilde{r}_j - r_f)\right] \le 0 \quad \forall j = 1, 2, \dots J$$

$$\Leftrightarrow \quad u'(w_0(1+r_f))\, E(\tilde{r}_j - r_f) \le 0 \quad \forall j = 1, 2, \cdots J$$

$$\Leftrightarrow \quad E(\tilde{r}_j - r_f) \le 0 \quad \forall j = 1, 2, \cdots J$$

i.e. none of the risky assets have a strictly positive risk premium.

Note: If $\exists j'$ s.t. $E(\tilde{r}_{j'} - r_f) > 0$

then $\exists j$ s.t. $\alpha_j > 0$ It is possible $j \ne j'$ for $E(\tilde{r}_j) > E(\tilde{r}_{j'})$. $\alpha_{j'} = 0$

Proposition (summary of above) An individual will take risky investments if and only if the expected return on at least one risky asset exceeds the riskless asset.

Motivation for two-fund separation

Let S be the total amount of initial wealth invested in the risky assets and θ_j be the proportion of S that invested in asset j.

then $\sum_{j=1}^{J} \theta_j = 1$. The uncertain end-of-period wealth:

$$\tilde{w} = (w_0 - S)(1+r_f) + \sum_{j=1}^{J} \theta_j S(\tilde{r}_j + 1)$$

$$= w_0(1+r_f) + S\sum_{j=1}^{J} \theta_j (\tilde{r}_j - r_f) = \underline{w_0(1+r_f) + S(\tilde{r} - r_f)}$$

where $\tilde{r} = \sum_{j=1}^{J} \theta_j \tilde{r}_j$

Remark. If an individual always choose to hold the same portfolio of risky assets and only change the mix between the portfolio and riskless assets for different level of initial wealth, then the comparative statics for two assets will be valid.

Remark: If the optimal portfolio composition $\theta^* = (\theta_1^*, \theta_2^*, \cdots, \theta_J^*)$ is unchanged over different levels of initial wealth, then the individual's optimal portfolios for different levels of initial wealth are always linear combinations of the riskless asset and a risky asset mutual fund. (Two-fund separation)

$$w^* = (w_0 - S^*)(1+r_f) + S^*\left(\sum_{j=1}^{J} \theta_j^* \tilde{r}_j + 1\right)$$

Theorem (Cass and Stiglitz, 1970)

A necessary and sufficient condition on utility function for two-fund separation is that: to ensure strict concavity and increasing

$$u'(z) = (A+Bz)^C \quad \text{where } B, C > 0, \ z \geq \max\{0, -\tfrac{A}{B}\}$$

$$\text{or } A>0, \ B<0, \ C>0, \ 0 \leq z < -\tfrac{A}{B}$$

$$\text{or } u'(z) = Ae^{Bz} \quad \text{where } A>0, \ B<0, \ z \geq 0$$

Proof. Necessity is omitted.

Sufficiency of the case $u'(z) = (A+Bz)^c$

let S. $\theta = (\theta_1, \theta_2, \cdots \theta_J) = \theta$ be the optimal solution for w_0

. Individual's choice problem:

$$\max_{S,\theta} E(u(\tilde{w})) \quad \text{s.t.} \sum_{j=1}^{J} \theta_j = 1$$

where $\tilde{w} = w_0(1+r_f) + S\left(\sum_{j=1}^{J} \theta_j \tilde{r}_j - r_f\right)$

F.O.C $\quad E\left(u'(\tilde{w})\left(\sum_{j=1}^{J}\theta_j \tilde{r}_j - r_f\right)\right) = 0$

$$E\left(u'(\tilde{w})\, S\,(\tilde{r}_j - r_f)\right) = 0 \qquad j = 1, 2, \cdots J$$

$$\sum_{j=1}^{J} \theta_j = 1$$

For interior solution.

$$\circledast \quad E\left(u'(\tilde{w})(\tilde{r}_j - r_f)\right) = 0 \qquad j = 1, 2, \cdots J$$

$$\sum_{j=1}^{J} \theta_j = 1 \qquad\qquad S' = \frac{A + Bw_0'(r_f+1)}{A + Bw_0(r_f+1)} S$$

Let $u'(z) = (A+Bz)^c$, $w_0 \neq w_0'$. Set $\theta_j' = \theta_j$. ✓

$$\tilde{w}' = w_0'(1+r_f) + S'\left(\sum_{j=1}^{J}\theta_j'\tilde{r}_j - r_f\right)$$

$$= w_0'(1+r_f) + S'\left(\sum_{j=1}^{J}\theta_j \tilde{r}_j - r_f\right)$$

$$A + B\tilde{w}' = A + Bw_0'(1+r_f) + B\left(\frac{A+Bw_0'(1+r_f)}{A+Bw_0(1+r_f)} \cdot S\right)\left(\sum_{j=1}^{J}\theta_j\tilde{r}_j - r_f\right)$$

$$= \left(A + Bw_0'(1+r_f)\right)\left[1 + \frac{BS}{A+Bw_0(1+r_f)}\left(\sum_{j=1}^{J}\theta_j\tilde{r}_j - r_f\right)\right]$$

$$= \frac{A+Bw_0'(1+r_f)}{A+Bw_0(1+r_f)}\left[A + Bw_0(1+r_f) + BS\left(\sum_{j=1}^{J}\theta_j\tilde{r}_j - r_f\right)\right]$$

$$= \frac{A+Bw_0'(1+r_f)}{A+Bw_0(1+r_f)}\left[A + B\tilde{w}\right]$$

$$\mu'(\tilde{w}') = \left(\frac{A+Bw_0'(1+r_f)}{A+Bw_0(1+r_f)}\right)^c [A+B\tilde{w}]^c = \left(\frac{A+Bw_0'(1+r_f)}{A+Bw_0(1+r_f)}\right)^c \mu'(\tilde{w})$$

Since $E(\mu'(\tilde{w})(\tilde{r}_j - r_f)) = 0.$ $\forall j = 1, 2 \dots J$

then $E(\mu'(\tilde{w}')(\tilde{r}_j - r_f)) = 0$ $\forall j = 1, 2 \dots J$ $\square$

Lec 4 Stochastic Dominance

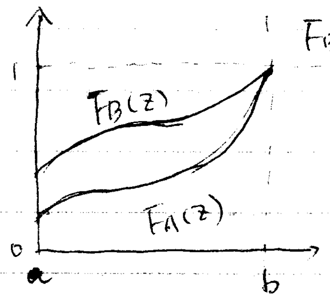
§1 First degree stochastic dominance

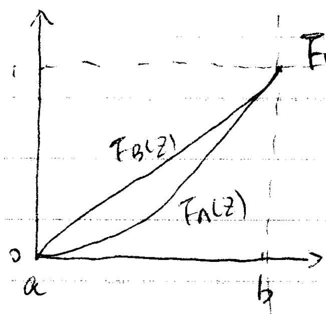
Def For risky assets A, B. A first degree stochastically dominates B, $A \underset{FSD}{\succcurlyeq} B$ if all individuals with non-decreasing utility function either prefer A to B or are indifferent with A and B

Def The distribution $F(\cdot)$ first ~~degree~~ order stochastically dominates $G(\cdot)$ if for every non-decreasing function $u(\cdot): \mathbb{R} \to \mathbb{R}$ we have

$$\int u(x)\,dF(x) \geq \int u(x)\,dG(x)$$

Prop $A \underset{FSD}{\succcurlyeq} B$ iff $F_A(z) \leq F_B(z)$ $\forall z \in [a,b] \subset \mathbb{R}$





$F_B(a) \neq F_B(a)$
$\neq 0$ $\neq 0$

discrete distributed random variable

continuous distributed random variable

eg: if considering rate of return.
$[a,b] = [-1,1]$

Remark $A \underset{FSD}{\succcurlyeq} B$ does not mean that A always has a higher realized rate of return.

eg. five potential outcomes with equal probability

	A	B
$\frac{1}{5}$	$\frac{1}{2}$	1
$\frac{1}{5}$	$\frac{1}{2}$	0
$\frac{1}{5}$	$\frac{1}{2}$	0
$\frac{1}{5}$	1	0
$\frac{1}{5}$	1	0

z	0	$\frac{1}{2}$	1
$F_A(z)$	0	$\frac{3}{5}$	1
$F_B(z)$	$\frac{4}{5}$	$\frac{4}{5}$	1

Recall $A \underset{FSD}{\succsim} B$ iff $F_A(z) \le F_B(z)$ $\forall z \in [a,b]$

proof (Sufficiency) Suppose $F_A(z) \le F_B(z)$ $\forall z \in [a,b]$.

Want: $A \underset{FSD}{\succsim} B \Leftrightarrow E(u(w_0(1+\tilde{r}_A))) \ge E(u(w_0(1+\tilde{r}_B)))$ $\forall$ nondecreasing $u(\cdot)$

$$\Leftrightarrow \int_{[a,b]} u(w_0(1+z))\, dF_A(z) \ge \int_{[a,b]} u(w_0(1+z))\, dF_B(z)$$

$$\Leftrightarrow \int_{[a,b]} u(w_0(1+z))\, d\big(F_A(z) - F_B(z)\big) \ge 0 \quad \circledast$$

$$\int_{[a,b]} u(w_0(1+z))\, d\big(F_A(z) - F_B(z)\big) = \Big[u(w_0(1+z))\underbrace{\cancel{(F_A(z) - F_B(z))}}_{}^{\,0}\Big]_a^b - \int_a^b u'(w_0(1+z))(F_A(z) - F_B(z))\, dz$$

$$= -\int_a^b w_0 \underbrace{u'(w_0(1+z))}_{\ge 0} \underbrace{(F_A(z) - F_B(z))}_{\le 0}\, dz \ge 0$$

So $\circledast$ is verified.

(Necessity) Suppose $A \underset{FSD}{\succsim} B$. Want: $F_A(z) \le F_B(z)$ $\forall z \in [a,b]$

Suppose $F_A(x) > F_B(x)$ for some $x \in [a,b]$

Since $\underset{\text{the c.d.f}}{F}$ is increasing and right-continuous then $\exists$ interval $[x, c] \subset [a,b)$ s.t. $F_A(z) > F_B(z)$ $\forall z \in [x,c]$

Let $s(y) = F_A(y) - F_B(y)$. then $s(y) > 0$ $\forall y \in [x,c]$

$$\int_{[a,b]} u((1+z)w_0)\, d\big(F_A(z) - F_B(z)\big)$$

$$= \Big[u((1+z)w_0)\,\cancel{(F_A(z) - F_B(z))}^{\,0}\Big]_a^b - \int_{[a,b]} F_A(z) - F_B(z)\, d\,u((1+z)w_0)$$

$$= -\left(\int_a^x + \int_x^c + \int_c^b\right) s(z)\, d\,u((1+z)w_0) \qquad ①$$

Note Let $u((1+z)w_0)$ be constant on $[a,x]$ and $[c,b]$. then by ①

$$\int_{[a,b]} u((1+z)w_0)\, d\big(F_A(z) - F_B(z)\big) = -\int_x^c s(z)\, d\,u((1+z)w_0)$$

Construct $\mu((1+z)w_0) = \begin{cases} c & \text{if } z \geq c \\ z & \text{if } x \leq z \leq c \\ x & \text{if } z \leq x \end{cases}$

the $\int_{[a,b]} \mu((1+z)w_0)\, d\left(F_A(z) - F_B(z)\right) = -\int_x^c S(z)\, d\,u((1+z)w_0)$

$= -\int_x^c S(z)\, dz < 0$ since $S(y) > 0 \quad \forall y \in [x, c]$

Contradiction to $A \underset{FSD}{\succeq} B$. $\square$

Prop $A \underset{FSD}{\succeq} B$ iff $\tilde{r}_A \overset{d}{=} \underbrace{\tilde{r}_B + \tilde{\alpha}}$ with $\tilde{\alpha} \geq 0$

+ve random variable

$\Rightarrow$ $A \underset{FSD}{\succeq} B \iff F_A(z) \leq F_B(z) \ \forall z \in [a,b] \iff \tilde{r}_A \overset{d}{=} \tilde{r}_B + \tilde{\alpha}, \ \tilde{\alpha} \geq 0$

Remark Choose $u((1+z)w_0) = z$. then from $A \underset{FSD}{\succeq} B$.

$\int z\, dF_A(z) = \underbrace{E(\tilde{r}_A) \geq E(\tilde{r}_B)} = \int z\, dF_B(z)$

asset A has as least as high expected rate of return as B.

... The converse is NOT true.

§2. Second degree schochastic dominance

Def Risky asset A dominates risky asset B in the sense of second degree stochastic dominance, denoted by $A \underset{SSD}{\succeq} B$ if all risky averse individuals either prefer A to B or are indifferent with A and B.

Def For any two distributions $F_A(\cdot)$ and $F_B(\cdot)$, $F_A(\cdot)$ second order stochastically dominates $F_B(\cdot)$ if for every concave function $\mu(\cdot): \mathbb{R} \to \mathbb{R}$ we have $\int \mu(x)\, dF_A(x) \geq \int \mu(x)\, dF_B(x)$.

Remark risk averse individuals may have utility functions that are not monotonically increasing i.e FSD and SSD

Prop Assume that all risk averse individuals have utility functions with continuous first derivatives, then

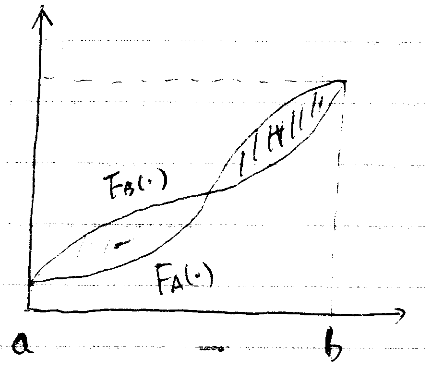
$$A \underset{SSD}{\succsim} B \quad \text{iff} \quad E(\tilde{r}_A) = E(\tilde{r}_B) \quad \text{and} \quad S(y) = \int_a^y F_A(z) - F_B(z)\, dz \le 0 \quad \forall y \in [a, b]$$

Remark $S(a) = 0$. If $E(\tilde{r}_A) = E(\tilde{r}_B)$ then $S(b) = 0$

$$0 = \underline{E(\tilde{r}_A) - E(\tilde{r}_B)} = \int_{[a,b]} z \, d\left(F_A(z) - F_B(z)\right)$$

$$= \Big[z\left(F_A(z) - F_B(z)\right)\Big]_a^b - \int_{[a,b]} F_A(z) - F_B(z)\, dz$$

$$= -\int_{[a,b]} F_A(z) - F_B(z)\, dz = \underline{-S(b)} \Rightarrow S(b) = 0$$



second degree stochastic dominates

$S(b) = 0$ basically says that the absolute value of areas of (+) and (−) must be equal.

proof (Sufficiency) Suppose $S(y) \le 0 \quad \forall y \in [a,b]$, $S(b) = 0 = E(\tilde{r}_B) - E(\tilde{r}_A)$

Want to show: $A \underset{SSD}{\succsim} B \iff E\left(\mu((1+\tilde{r}_A)w_0)\right) \ge E\left(\mu((1+\tilde{r}_B)w_0)\right)$

for any concave utility $\mu(\cdot)$

$$E\left(\mu((1+\tilde{r}_A)w_0)\right) - E\left(\mu((1+\tilde{r}_B)w_0)\right) = \int_a^b \mu((1+z)w_0)\, d\left(F_A(z) - F_B(z)\right)$$

$$= -\int_a^b F_A(z) - F_B(z)\, d\,\mu((1+z)w_0) = -w_0 \int_a^b \mu'((1+z)w_0)\, dS(z)$$

$$= -w_0 \Big[\mu'((1+z)w_0)\, S(z)\Big]_a^b + w_0 \int_a^b S(z)\, d\mu'((1+z)w_0)$$

$$= w_0 \int_a^b S(z)\, d\mu'((1+z)w_0) \ge 0 \quad \text{Since } \mu'(\cdot) \text{ nonincreasing}, \; S(z) \le 0$$

(Necessity) Suppose $A \underset{SSD}{\geq} B$. Want $E(\tilde{r}_A) = E(\tilde{r}_B)$, $S(y) \leq 0 \ \forall y \in [a,b]$

Firstly note $\mu(z) = z$ and $\mu(z) = -z$ are both concave. then.

by $A \underset{SSD}{\geq} B$. $\int_{[a,b]} (1+z) w_0 \, dF_A(z) \geq \int_{[a,b]} (1+z) w_0 \, dF_B(z)$

$$\Rightarrow \int_{[a,b]} z \, dF_A(z) \geq \int_{[a,b]} z \, dF_B(z)$$

$$\Rightarrow E(\tilde{r}_A) \geq E(\tilde{r}_B) \quad ①$$

$$\int_{[a,b]} -(1+z) w_0 \, dF_A(z) \geq \int_{[a,b]} -(1+z) w_0 \, dF_B(z)$$

$$\Rightarrow \int_{[a,b]} z \, dF_B(z) \geq \int_{[a,b]} z \, dF_A(z) \Rightarrow E(\tilde{r}_B) \geq E(\tilde{r}_A) \quad ②$$

by ① and ②. $E(\tilde{r}_A) = E(\tilde{r}_B)$.

Suppose $\exists \ x \in [a,b)$ s.t $S(x) > 0$. by continuity of $S(\cdot)$.

$\exists$ interval $[\xi_1, \xi_2]$ with $\xi_1 \neq \xi_2$, containing x . s.t

$S(z) > 0 \ \forall z \in [\xi_1, \xi_2]$

$$E(\mu((1+\tilde{r}_A) w_0)) - E(\mu((1+\tilde{r}_B) w_0)) = w_0 \left(\int_a^{\xi_1} + \int_{\xi_1}^{\xi_2} + \int_{\xi_2}^{b} \right) S(z) \, d\mu'((1+z) w_0)$$

Construct utility function ~~as~~ satisfying

$$\mu'((1+z) w_0) = \begin{cases} -\xi_1 & z \leq \xi_1 \\ -z & \xi_1 < z < \xi_2 \\ -\xi_2 & z \geq \xi_2 \end{cases}$$

Where $\mu((1+z) w_0) = \int_a^z \mu'((1+t) w_0) \, dt$ is continuously differentiable and concave

then $E(\mu((1+\tilde{r}_A) w_0)) - E(\mu((1+\tilde{r}_B) w_0)) = w_0 \int_{\xi_1}^{\xi_2} S(z) \, d\mu'((1+z) w_0)$

$$= w_0 \int_{\xi_1}^{\xi_2} -S(z) \, dz < 0 \ . \ \text{Contradiction}$$

Prop (Rothschild and Stiglitz, 1970)

$A \succeq_{SSD} B$ if and only if $\tilde{r}_B \overset{d}{=} \tilde{r}_A + \tilde{\varepsilon}$ with $E(\tilde{\varepsilon} \mid \tilde{r}_A) = 0$.

We call risky asset B a mean-preserving spread of A.

Remark $A \succeq_{SSD} B \Leftrightarrow \tilde{r}_B \overset{d}{=} \tilde{r}_A + \tilde{\varepsilon}$ with $E(\tilde{\varepsilon} \mid \tilde{r}_A) = 0$

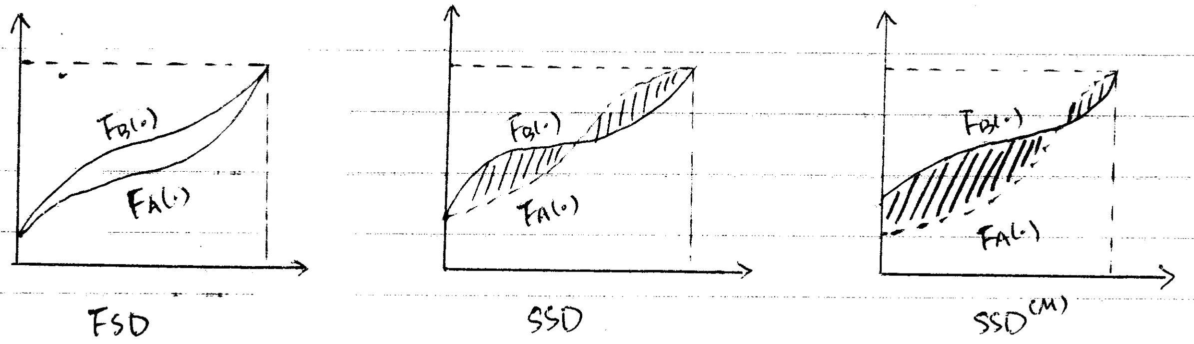
B is more risky than A

$\Downarrow$ $\not\Uparrow$ Risk is more than just variance.

$var(\tilde{r}_B) = var(\tilde{r}_A) + var(\tilde{\varepsilon}) \geq var(\tilde{r}_A)$

where $cov(\tilde{r}_A, \tilde{\varepsilon}) = E((\tilde{r}_A - E(\tilde{r}_A))\tilde{\varepsilon}) = E(\tilde{r}_A \tilde{\varepsilon}) - E(\tilde{r}_A)E(\tilde{\varepsilon}) = 0$

§3 Second degree stochastic monotonic dominance $\succeq^{M}_{SSD}$



Def Risky asset A dominates B in the sense of second degree stochastic monotonic dominance, denoted by $\succeq^{M}_{SSD}$ if all individuals who are risk averse and having nondecreasing utility functions prefer A to B.

Prop $A \succeq^{M}_{SSD} B \Leftrightarrow E(\tilde{r}_A) \geq E(\tilde{r}_B)$ and $S(z) \leq 0 \quad \forall z \in [a, b]$

$\Leftrightarrow \tilde{r}_B \overset{d}{=} \tilde{r}_A + \tilde{\varepsilon}$ with $E(\tilde{\varepsilon} \mid \tilde{r}_A) \leq 0$

Lec 5 Mean-variance efficient frontier

§ Motivation for mean-variance analysis

1° $\succsim_{SSD} \Rightarrow E(\tilde{r}_A) = E(\tilde{r}_B),\ \operatorname{var}(\tilde{r}_B) \geq \operatorname{var}(\tilde{r}_A) \quad \forall\, \tilde{r}_B$ (portfolio)

i.e. a portfolio of assets second degree stochastically dominates all portfolios that have the same expected rate of return ⇒ it has the minimal variance among all.

2° For arbitrary distributions and utility functions, expected utility cannot be defined over just the expected returns and variances.

$$u(\tilde{\omega}) = u(E(\tilde{\omega})) + u'(E(\tilde{\omega}))(\tilde{\omega} - E(\tilde{\omega})) + \frac{1}{2}u''(E(\tilde{\omega}))(\tilde{\omega} - E(\tilde{\omega}))^2 + R_3$$

by assuming Taylor series converges and expectation and summation operations are interchangable.

higher order term cannot be ignored in general

$$E(u(\tilde{\omega})) = u(E(\tilde{\omega})) + u'(E(\tilde{\omega}))\underbrace{E(\tilde{\omega} - E(\tilde{\omega}))}_{0} + \frac{1}{2}u''(E(\tilde{\omega}))\,E\left((\tilde{\omega} - E(\tilde{\omega}))^2\right) + E(R_3)$$

$$= u(E(\tilde{\omega})) + \frac{1}{2}u''(E(\tilde{\omega}))\,\sigma^2(\tilde{\omega}) + E(R_3)$$

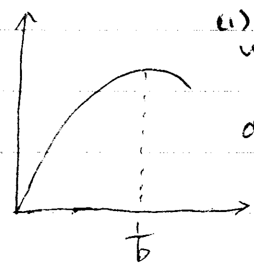
needs special assumptions on distributions and/or utility functions

3°. For arbitrary distributions, assuming quadratic utility

$$u(z) = z - \frac{1}{2}bz^2, \quad b > 0$$

$E(R_3) = 0$ in this case. then

$$E(u(\tilde{\omega})) = E(\tilde{\omega}) - \frac{1}{2}b\left[E(\tilde{\omega})^2 + \sigma^2(\tilde{\omega})\right]$$



(1) when $z > \frac{1}{b}$ decreasing utility function

(2) increasing absolute risk aversion, i.e. risky assets are inferior goods.

4° For arbitrary preferences, assuming the rate of return on risky assets are multivariate normally distributed.

- under normality, the third and higher order moments in $E(R_3)$ can be expressed as functions of first two moments.

$$E[X^p] = \begin{cases} 0 & \text{if } p \text{ is odd} \\ \sigma^p (p-1)!! & \text{if } p \text{ is even} \end{cases}$$

where $n!!$ denotes the double factorial, i.e. the product of all numbers from n to 1 that have the same parity with n

- normal distributions are also stable under addition.

$$X + Y \sim N(\mu_X + \mu_Y, \sigma_X^2 + \sigma_Y^2) \quad \text{when } X, Y \text{ independent}$$

when X, Y jointly normally distributed, $\sigma_{X+Y} = \sqrt{\sigma_X^2 + \sigma_Y^2 + 2\rho\sigma_X\sigma_Y}$

- Prop: If $\tilde{r}_A \sim N(\mu, \sigma_A^2)$, $\tilde{r}_B \sim N(\mu, \sigma_B^2)$ with $\sigma_A^2 < \sigma_B^2$, then

$$A \underset{SSD}{\succsim} B$$

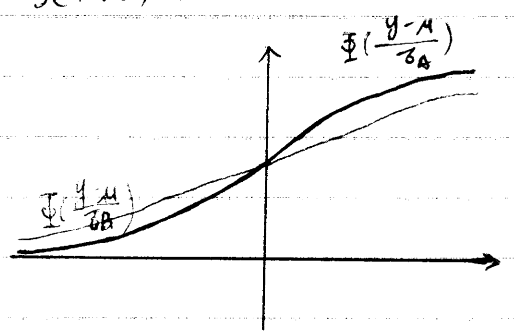
proof: Consider $S(y) = \int_{-\infty}^{y} F_A(z) - F_B(z)\, dz$.

$$0 = \mu - \mu = E(\tilde{r}_A) - E(\tilde{r}_B)$$

$$= \int_{-\infty}^{\infty} z\, dF_A(z) - \int_{-\infty}^{\infty} z\, dF_B(z) = \left[[z F_A(z)]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} F_A(z)\,dz \right] - \left[[z F_B(z)]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} F_B(z)\,dz \right]$$

$$= -\int_{-\infty}^{\infty} F_A(z) - F_B(z)\,dz = -S(+\infty) \implies S(+\infty) = 0$$

Also note $\frac{\tilde{r}_A - \mu}{\sigma_A}, \frac{\tilde{r}_B - \mu}{\sigma_B} \sim N(0,1)$



$$F_A(y) = \int_{-\infty}^{y} \frac{1}{\sqrt{2\pi}\sigma_A} \exp\left(-\frac{(x-\mu)^2}{2\sigma_A^2}\right) dx$$

$$= \int_{-\infty}^{\frac{y-\mu}{\sigma_A}} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}z^2\right) dz = \Phi\left(\frac{y-\mu}{\sigma_A}\right)$$

$z = \frac{x-\mu}{\sigma_A}$

$F_B(y) = \Phi\left(\frac{y-\mu}{\sigma_B}\right) \implies S'(y) \begin{cases} > 0 & \text{if } y > \mu \\ < 0 & \text{if } y < \mu \end{cases}$ Note $0 = S(+\infty) = S(-\infty)$ then $S(y) \le 0 \ \forall y$

Disadvantages 1° The normal distribution is unbounded from below. ie it can take arbitrarily negative value with positive probability. which is inconsistent with limited liability and with economic theory

2° utility functions like $u(z) = \ln(z)$ cannot be used.

5° other motivations:

easy: analytically tractable

useful: richness of empirical implications

§2. Preliminary Settings:

1° The market is frictionless:

no transaction costs, no taxes, no short-selling and borrowing restrictions

borrowing rate = lending rate.

2°. There are $N \geq 2$ risky assets on the market.

3°. Rates of return on these assets have finite variance and unequal expectations: $\tilde{r} = (\tilde{r}_1, \tilde{r}_2, \dots, \tilde{r}_N)$, $e = E(\tilde{r}) = (E(\tilde{r}_1), \dots, E(\tilde{r}_N))$

$$V = E((\tilde{r}-e)(\tilde{r}-e)^T) \text{ : variance - covariance matrix}$$

assume asset returns are linearly independent, then

V is positive definite and non-singular.

4° A portfolio is a vector $\theta = [\theta_1, \theta_2, \dots, \theta_N]^T$ with θ_j be the proportion of wealth invested in risky asset J.

For portfolios θ that consists of only risky assets, we have $\theta^T \mathbb{1} = 1$

§3 Mean-variance efficient frontier without riskless assets

Def A portfolio θ_p is a frontier portfolio if it has the minimum variance among portfolios that have the same expected return.

θ_p is the solution to the quardratic program

$$\min_{\{\theta\}} \frac{1}{2}\theta^T V \theta \quad \text{s.t.} \quad \theta^T e = E(\tilde{r}_p), \ \theta^T \mathbb{1} = 1$$

Remark: Short sale permits negative portfolio weights and weights > 1.
i.e. the range of expected return on feasible portfolios is unbounded

$$\mathcal{L} = \frac{1}{2}\theta^T V \theta + \lambda\left(E(\tilde{r}_p) - \theta^T e\right) + \gamma(1 - \theta^T \mathbb{1})$$

F.O.C: $V\theta_p - \lambda e - \gamma \mathbb{1} = 0$ ① Since V is P.D FOCs are sufficient to ensure the global minima.

$E(\tilde{r}_p) = \theta_p^T e$ ② $\theta_p^T \mathbb{1} = 1$ ③

① $\Rightarrow$ $\theta_p = V^{-1}(\lambda e + \gamma \mathbb{1}) = \lambda(V^{-1}e) + \gamma(V^{-1}\mathbb{1})$ ... ⑥

By ②. $E(\tilde{r}_p) = \lambda(e^T V^{-1} e) + \gamma(\mathbb{1}^T V^{-1} e)$ ... ④

③ $1 = \lambda(e^T V^{-1}\mathbb{1}) + \gamma(\mathbb{1}^T V^{-1}\mathbb{1})$ ... ⑤

Solving ④ and ⑤ for λ, γ,

multiplying ④ and ⑤ by $\mathbb{1}^T V^{-1}\mathbb{1}$ and $\mathbb{1}^T V^{-1} e$ resp, taking subtraction

$$E(\tilde{r}_p)(\mathbb{1}^T V^{-1}\mathbb{1}) - (\mathbb{1}^T V^{-1} e) = \lambda\left[(e^T V^{-1} e)(\mathbb{1}^T V^{-1}\mathbb{1}) - (e^T V^{-1}\mathbb{1})^2\right]$$

$$\Rightarrow \lambda = \left(E(\tilde{r}_p)(\mathbb{1}^T V^{-1}\mathbb{1}) - (\mathbb{1}^T V^{-1} e)\right) \Big/ \underbrace{\left[(e^T V^{-1} e)(\mathbb{1}^T V^{-1}\mathbb{1}) - (e^T V^{-1}\mathbb{1})^2\right]}_{D = BC - A^2}$$

Similarly.

$$\gamma = \left(E(\tilde{r}_p)\underbrace{(e^T V^{-1}\mathbb{1})}_{A} - (e^T V^{-1} e)\right) \Big/ \left[(e^T V^{-1}\mathbb{1})^2 - \underbrace{(\mathbb{1}^T V^{-1}\mathbb{1})}_{C}\underbrace{(e^T V^{-1} e)}_{B}\right]$$

$$\Rightarrow \lambda = \frac{E(\tilde{r}_p)C - A}{D}, \quad \gamma = \frac{E(\tilde{r}_p)A - B}{-D}$$

A. B. C. D: functions of mean and variance covariance matrix.

by ⑥

$$\theta_p = \underbrace{\frac{1}{D}\{B(V^{-1}\mathbb{1}) - A(V^{-1}e)\}}_{g} + \underbrace{\frac{1}{D}\{C(V^{-1}e) + A(V^{-1}\mathbb{1})\}}_{h} E(\tilde{r}_p)$$

$$= g + hE(\tilde{r}_p) \quad ⊛$$

Then any portfolio with presentation of ⊛ is a frontier portfolio.

Def The set of all frontier portfolios is call portfolio frontier.

Note g is a frontier portfolio with 0 expected rate of return.

$g+h$ - - - - - - - - - - - - - - 1 - - - - - - - - -

For any frontier portfolio θ_q with expected rate of return $E(\tilde{r}_q)$

i.e. $\theta_q = g + hE(\tilde{r}_q)$

then $\theta_q = g(1-E(\tilde{r}_q)) + (g+h)E(\tilde{r}_q)$ Combination of g. $g+h$

Remark: The entire portfolio frontier can be generated by combination of g and $g+h$.

Prop The entire portfolio frontier can be generated by any two distinct frontier portfolios.

proof: Let $\theta_{p_1}, \theta_{p_2}$ be two distinct portfolios. so $E(\tilde{r}_{p_1}) \neq E(\tilde{r}_{p_2})$

Let θ_q be any frontier portfolio with $E(\tilde{r}_q)$. then $\exists\ \alpha \in \mathbb{R}$ st.

$$\alpha E(\tilde{r}_{p_1}) + (1-\alpha)E(\tilde{r}_{p_2}) = E(\tilde{r}_q).$$

$$\begin{aligned}\alpha\theta_{p_1} + (1-\alpha)\theta_{p_2} &= \alpha(g + hE(\tilde{r}_{p_1})) + (1-\alpha)(g + hE(\tilde{r}_{p_2})) \\ &= g + h[\alpha E(\tilde{r}_{p_1}) + (1-\alpha)E(\tilde{r}_{p})] \\ &= g + hE(\tilde{r}_q) = \theta_q\end{aligned}$$

• Picture of portfolio frontier in the $\sigma(\tilde{r}) - E(\tilde{r})$ space

$$\begin{aligned}\mathrm{cov}(\tilde{r}_p, \tilde{r}_q) &= \theta_p^T V \theta_q = (g + E(\tilde{r}_p)h)^T V(g + hE(\tilde{r}_q)) \\ &= \underline{g^T V g} + \underline{g^T V h}\, E(\tilde{r}_q) + \underline{h^T V g}\, E(\tilde{r}_p) + \underline{h^T V h}\, E(\tilde{r}_p)E(\tilde{r}_q)\end{aligned}$$

Note $g^T V g = \frac{1}{D^2}\{B(\mathbb{1}^T V^{-1}) - A(e^T V^{-1})\} V \{B(V^{-1}\mathbb{1}) - A(V^{-1}e)\}$

$$= \frac{1}{D^2}\{B\mathbb{1}^T - Ae^T\}\{B(V^{-1}\mathbb{1}) - A(V^{-1}e)\}$$

$$= \frac{1}{D^2}\{B^2C - 2A^2B + A^2B\} = \frac{B}{D^2}\{BC - A^2\} = \frac{B}{D}$$

$$g^T V h = \frac{1}{D^2}\{B\mathbb{1}^T - Ae^T\}\{C(V^{-1}e) - A(V^{-1}\mathbb{1})\}$$

$$= \frac{1}{D^2}\{ABC - ABC - ABC + A^3\} = \frac{1}{D^2}(A^3 - ABC) = -\frac{A}{D}$$

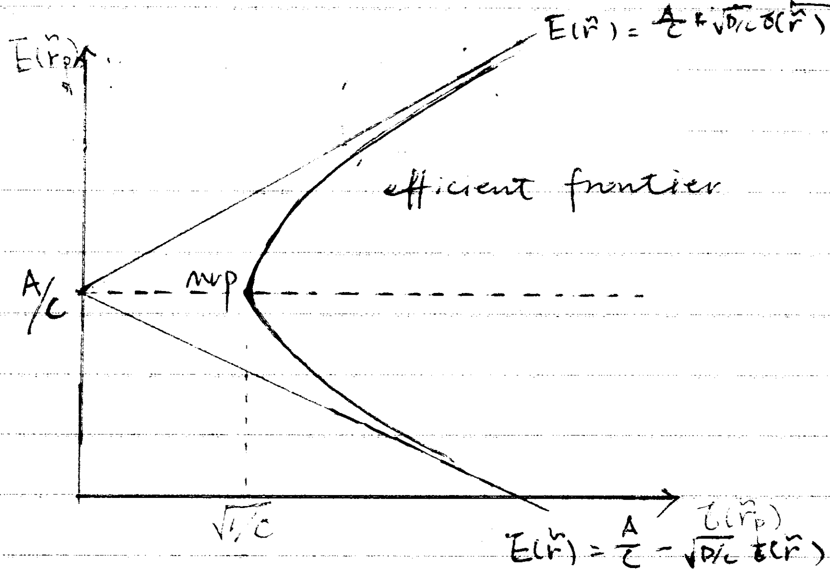
$$h^T V h = \frac{1}{D^2}\{C(e^T V^{-1}) - A(\mathbb{1}^T V^{-1})\}\{Ce - A\mathbb{1}\}$$

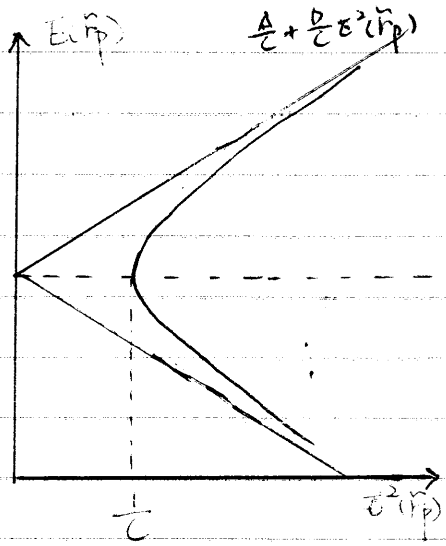
$$= \frac{1}{D^2}\{C^2B - A^2C - A^2C + A^2C\} = \frac{1}{D^2}\{BC^2 - A^2C\} = \frac{C}{D}$$

$$\Rightarrow \operatorname{cov}(\tilde{r}_p, \tilde{r}_q) = \frac{B}{D} - \frac{A}{D}\left(E(\tilde{r}_q) + E(\tilde{r}_p)\right) + \frac{C}{D}E(\tilde{r}_p)E(\tilde{r}_q)$$

$$= \frac{C}{D}\left[E(\tilde{r}_q) - \frac{A}{C}\right]\left[E(\tilde{r}_p) - \frac{A}{C}\right] + \frac{1}{C}$$

$$\Rightarrow \sigma^2(\tilde{r}_p) = \frac{C}{D}\left[E(\tilde{r}_p) - \frac{A}{C}\right]^2 + \frac{1}{C} \Rightarrow \boxed{\frac{\sigma^2(\tilde{r}_p)}{1/C} - \frac{\left[E(\tilde{r}_p) - \frac{A}{C}\right]^2}{D/C^2} = 1} \quad (*)$$





- minimum variance portfolio (mvp)

$\sigma^2(\tilde{r}_p) = \frac{C}{D}\left[E(\tilde{r}_p) - \frac{A}{C}\right]^2 + \frac{1}{C}$ the minimum variance portfolio has mean $E(\tilde{r}) = \frac{A}{C}$, variance $\frac{1}{C}$. i.e. mvp $\sim \left(\frac{A}{C}, \frac{1}{C}\right)$

$$\operatorname{cov}(\tilde r_p, \tilde r_q) = \frac{C}{D}\left(E(\tilde r_p) - \frac{A}{C}\right)\left(E(\tilde r_q) - \frac{A}{C}\right) + \frac{1}{C}$$

then $\forall$ frontier portfolio θ_p. $\operatorname{cov}(\tilde r_p, \tilde r_{mvp}) = \operatorname{var}(\tilde r_{mvp}) = \frac{1}{C}$

- Efficient frontier and Black's separation theorem.

<u>Def</u> Effient portfolios are frontier portfolios with expected rates of return strictly higher ~~that~~ than that of MVP $= \frac{A}{C}$.

Inefficient portfolios are frontier portfolios that are not efficient.

Efficient frontier is the set of all efficient portfolios.

<u>Thm</u> (Black's separation theorem)

(a) Convex combination of two efficient portfolios is an efficient portfolio.

i.e. let θ', θ'' be two efficient portfolios, then $\forall \alpha \in (0,1)$ $\alpha\theta' + (1-\alpha)\theta''$ is an efficient portfolio.

(b) let θ', θ'' be two distinct efficient portfolios, then for any efficient portfolio θ, there exists a unique $\alpha \in \mathbb{R}$ s.t. $\theta = \alpha\theta' + (1-\alpha)\theta''$

<u>proof</u>. Note (b) is a direct corollary of the result that entire portfolio frontier can be generated by any two distinct frontier portfolios.

(a): We have $\theta' = g + h E(\tilde r_{\theta'})$, $\theta'' = g + h E(\tilde r_{\theta''})$. for $\alpha \in (0,1)$

$$\alpha\theta' + (1-\alpha)\theta'' = g + h\left[\alpha E(\tilde r_{\theta'}) + (1-\alpha) E(\tilde r_{\theta''})\right]$$

Since $E(\tilde r_{\theta'}) > \frac{A}{C}$. $E(\tilde r_{\theta''}) > \frac{A}{C}$ by efficiency. then

$$\alpha E(\tilde r_{\theta'}) + (1-\alpha) E(\tilde r_{\theta''}) > \alpha\frac{A}{C} + (1-\alpha)\frac{A}{C} = \frac{A}{C}.$$

hence $\alpha\theta' + (1-\alpha)\theta''$ is an efficient portfolio. ▯

<u>Prop</u> For any frontier portfolio θ_p except for MVP. there exists a unique frontier portfolio denoted by $ZC(p)$, which 0 covariance with θ_p.

proof $\operatorname{cov}(\tilde r_p, \tilde r_{zc(p)}) = \frac{C}{D}\left[E(\tilde r_p) - \frac{A}{C}\right]\left[E(\tilde r_{zc(p)}) - \frac{A}{C}\right] + \frac{1}{C} = 0$

$\Rightarrow \quad E(\tilde r_{zc(p)}) = \frac{A}{C} - \frac{D/C^2}{E(\tilde r_p) - \frac{A}{C}}$ and there is a unique frontier portfolio with expected rate of return. ◇

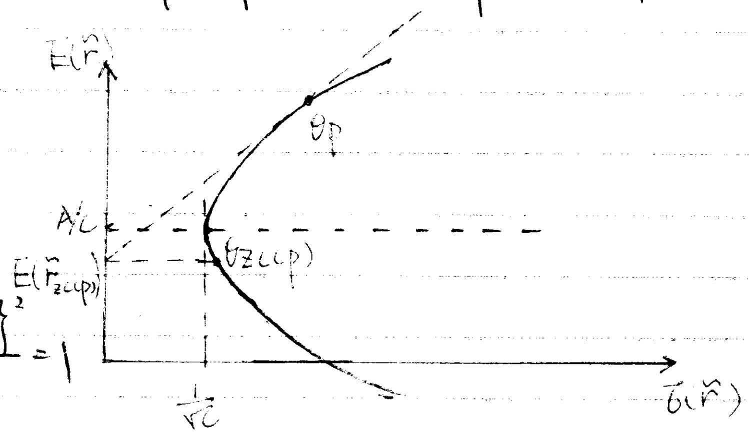
Remark by V is a positive definite matrix, $A, C, D > 0$.

then if θ_p is an efficient portfolio, then $\theta_{zc(p)}$ is inefficient

θ_p is an inefficient portfolio, $\theta_{zc(p)}$ is efficient.

Location of $zc(p)$:

in $E(\tilde r) - \sigma(\tilde r)$ space:



Recall: $\frac{\sigma^2(\tilde r_p)}{1/C} - \frac{\{E(\tilde r_p) - \frac{A}{C}\}^2}{D/C^2} = 1$

applying implicit function theorem,

$$\frac{dE(\tilde r_p)}{d\sigma(\tilde r_p)} = \frac{D\sigma(\tilde r_p)}{CE(\tilde r_p) - A} \quad \text{and tangent line is}$$

$$E(\tilde r) = E(\tilde r_p) + \frac{D\sigma(\tilde r_p)}{CE(\tilde r_p) - A}\left(\sigma(\tilde r) - \sigma(\tilde r_p)\right)$$

the intercept on $E(\tilde r)$ axis is $E(\tilde r_p) - \frac{D\sigma^2(\tilde r_p)}{CE(\tilde r_p) - A}$ ⊛

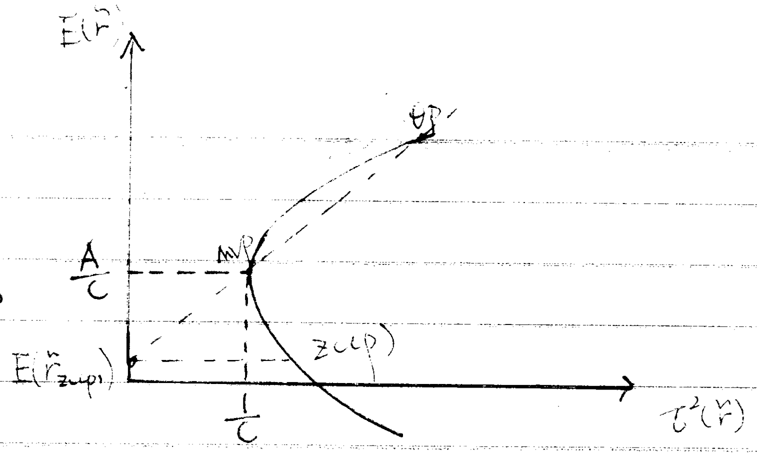
$$⊛ = \frac{1}{CE(\tilde r_p) - A}\left[CE^2(\tilde r_p) - AE(\tilde r_p) - D\sigma^2(\tilde r_p)\right]$$

$$= \frac{1}{E(\tilde r_p) - \frac{A}{C}}\left[E^2(\tilde r_p) - \frac{A}{C}E(\tilde r_p) - \frac{D}{C}\left(\frac{1}{C} + \frac{C}{D}\left(E(\tilde r_p) - \frac{A}{C}\right)^2\right)\right]$$

$$= \frac{1}{E(\tilde r_p) - \frac{A}{C}}\left[\frac{A}{C}E(\tilde r_p) - \frac{A^2}{C^2} - \frac{D}{C^2}\right] = \frac{A}{C} - \frac{D/C^2}{E(\tilde r_p) - \frac{A}{C}} = E(\tilde r_{zc(p)}).$$ ◇

in $E(\tilde{r})-\sigma^2(\tilde{r})$ space:

the line joining MVP and θ_p is:



$$E(\tilde{r})-\frac{A}{C}=\frac{E(\tilde{r}_p)-\frac{A}{C}}{\sigma^2(\tilde{r}_p)-\frac{1}{C}}\left[\sigma^2(\tilde{r})-\frac{1}{C}\right]$$. letting $\sigma^2(\tilde{r})=0$

$$E(\tilde{r})=\frac{A}{C}-\frac{1}{C}\,\frac{E(\tilde{r}_p)-\frac{A}{C}}{\sigma^2(\tilde{r}_p)-\frac{1}{C}}$$

$$=\frac{A}{C}-\frac{1}{C}\,\frac{E(\tilde{r}_p)-\frac{A}{C}}{\frac{C}{D}\left(E(\tilde{r}_p)-\frac{A}{C}\right)^2}=\frac{A}{C}-\frac{D/C^2}{E(\tilde{r}_p)-\frac{A}{C}}=E(\tilde{r}_{z(p)})$$ ◇

Prop. The expected rate of return on any portfolio θ_q (not necessarily on the frontier) can be written as a linear combination of the expected return on frontier portfolio $\theta_p \neq$ MVP and its zero covariance portfolio. (decomposition)

Proof Let θ_p be frontier portfolio other than MVP. Let θ_q be any portfolio not necessarily frontier portfolio.

$$\mathrm{cov}(\tilde{r}_p,\tilde{r}_q)=\theta_p^T V\theta_q$$

$$=\left(\lambda(V^{-1}e)+\gamma(V^{-1}\mathbb{1})\right)^T V\theta_q$$

$$=\left(\lambda e^T+\gamma\mathbb{1}^T\right)\theta_q=\lambda e^T\theta_q+\gamma=\lambda E(\tilde{r}_q)+\gamma$$

where $\lambda=\dfrac{CE(\tilde{r}_p)-A}{D}$, $\gamma=\dfrac{B-AE(\tilde{r}_p)}{D}$.

$$E(\tilde{r}_q)=\frac{1}{\lambda}\left(\mathrm{cov}(\tilde{r}_p,\tilde{r}_q)-\gamma\right)=\frac{D}{CE(\tilde{r}_p)-A}\left(\mathrm{cov}(\tilde{r}_p,\tilde{r}_q)-\frac{B-AE(\tilde{r}_p)}{D}\right)$$

$$=\frac{D}{CE(\tilde{r}_p)-A}\mathrm{cov}(\tilde{r}_p,\tilde{r}_q)+E(\tilde{r}_{z(p)})$$

$$= E(\tilde{r}_{zc(p)}) + \overbrace{\boxed{\frac{Cov(\tilde{r}_p, \tilde{r}_q)}{\sigma^2(\tilde{r}_p)}}}^{\hat{\beta}_{qp}} \left(\frac{C}{D} [E(\tilde{r}_p) - \frac{A}{C}]^2 + \frac{1}{C} \right) \cdot \frac{D/C}{E(\tilde{r}_p) - A/C}$$

$$= E(\tilde{r}_{zc(p)}) + \beta_{qp} \left[\left(E(\tilde{r}_p) - \frac{A}{C} \right) + \frac{D/C^2}{E(\tilde{r}_p) - \frac{A}{C}} \right]$$

$$= E(\tilde{r}_{zc(p)}) + \beta_{qp} \left(E(\tilde{r}_p) - E(\tilde{r}_{zc(p)}) \right)$$

$$= (1 - \beta_{qp}) E(\tilde{r}_{zc(p)}) + \beta_{qp} E(\tilde{r}_p)$$

Note $zc(p)$ is also a frontier portfolio, $zc(zc(p)) = \theta p$.

So $E(\tilde{r}_q) = (1 - \beta_{q\,zc(p)}) E(\tilde{r}_p) + \beta_{q\,zc(p)} E(\tilde{r}_{zc(p)})$

when $E(\tilde{r}_p) \neq E(\tilde{r}_{zc(p)})$, there exists a unique number α s.t

$$E(\tilde{r}_q) = \alpha E(\tilde{r}_p) + (1-\alpha) E(\tilde{r}_{zc(p)}).$$

$$\Rightarrow \quad 1 - \beta_{q\,zc(p)} = \beta_{qp}$$

$$\Rightarrow \quad E(\tilde{r}_q) = \beta_{q\,zc(p)} E(\tilde{r}_{zc(p)}) + \beta_{qp} E(\tilde{r}_p) .$$

linear decomposition

△ Mean-variance efficient frontier with riskless assets

- r_f: risk-free interest rate, $N \geq 2$ risky assets
- portfolio $\theta = [\theta_1, \theta_2, \cdots \theta_N]^T$ with θ_j be the propotion of wealth invested in risky asset j
- $1 - \theta^T \mathbb{1}$ is the proportion of wealth invested in riskless asset

$$\underbrace{E(\tilde{r})}_{\text{expected rate of return.}} = \tilde{E}\left(\frac{\tilde{w}}{w_0} - 1 \right) \underset{\uparrow}{=} \theta^T e + (1 - \theta^T \mathbb{1}) r_f$$

Individual's problem: $\min_{\{\theta\}} \frac{1}{2}\theta^T V \theta$ s.t. $\theta^T e + (1-\theta^T \mathbb{1}) r_f = E(\tilde{r}_p)$

θ_p is the Solution to the quadratic problem.

$$\mathcal{L} = \frac{1}{2}\theta^T V\theta + \lambda\left(E(\tilde{r}_p) - \theta^T e - (1-\theta^T\mathbb{1}) r_f\right)$$

FOC $\quad V\theta_p - \lambda e + \lambda r_f \mathbb{1} = 0 \Rightarrow \theta_p = V^{-1}(\lambda e - \lambda r_f \mathbb{1}) \cdots ①$

$$E(\tilde{r}_p) - \theta_p^T e - (1-\theta_p^T\mathbb{1}) r_f = 0 \cdots ②$$

taking ① into ②

$$\begin{aligned} E(\tilde{r}_p) - r_f &= \theta_p^T e - \theta_p^T(r_f \mathbb{1}) \\ &= \lambda(e^T - r_f\mathbb{1}^T)V^{-1}(e - r_f\mathbb{1}) \\ &= \lambda \underbrace{(e - r_f\mathbb{1})^T V^{-1}(e - r_f\mathbb{1})}_{H > 0 \text{ by positive definiteness}} = \lambda H \end{aligned}$$

$$\Rightarrow \lambda = H^{-1}\left(E(\tilde{r}_p) - r_f\right) = (B - 2Ar_f + Cr_f^2)^{-1}\left(E(\tilde{r}_p) - r_f\right)$$

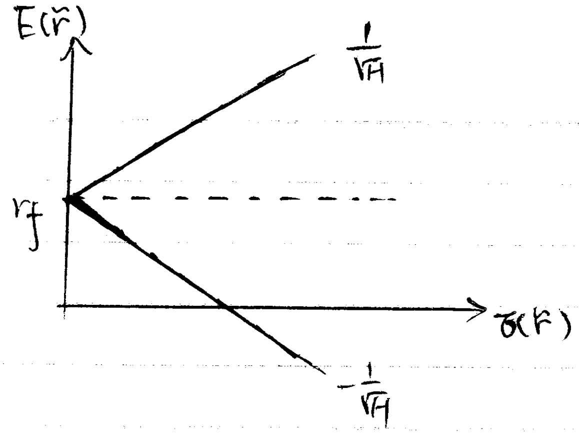
$$\begin{aligned} H &= e^T V^{-1} e - 2(e^T V^{-1}\mathbb{1}) r_f + r_f^2(\mathbb{1}^T V^{-1}\mathbb{1}) \\ &= B - 2Ar_f + Cr_f^2 \end{aligned}$$

$$\Rightarrow \theta_p = \lambda V^{-1}(e - r_f\mathbb{1}) = \frac{E(\tilde{r}_p) - r_f}{H} V^{-1}(e - r_f\mathbb{1}) \cdots ⊛$$

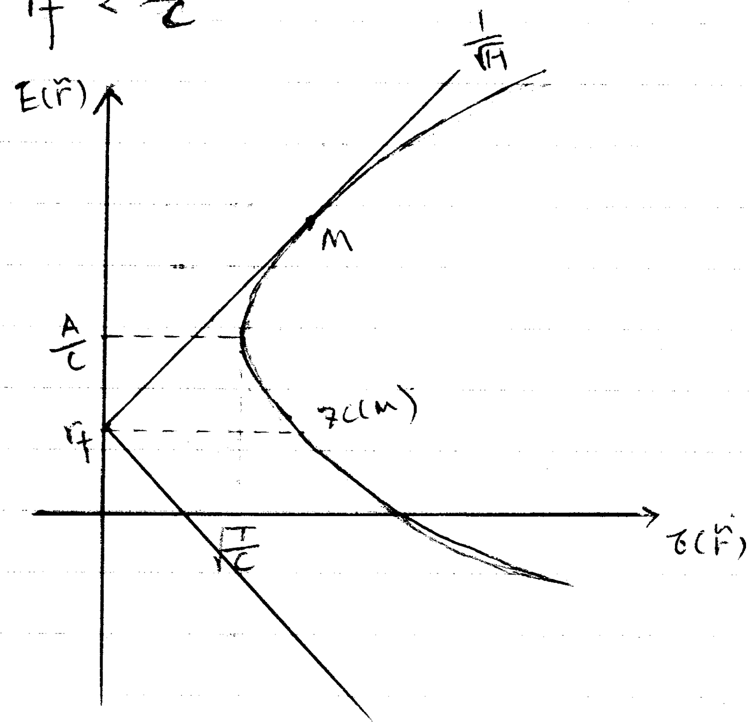
$$\underline{\sigma^2(\tilde{r}_p)} = \theta_p^T V \theta_p = \left(\frac{E(\tilde{r}_p) - r_f}{H}\right)^2 \underbrace{(e - r_f\mathbb{1})^T V^{-1}(e - r_f\mathbb{1})}_{H} = \underline{\frac{1}{H}\left(E(\tilde{r}_p) - r_f\right)^2} \cdots (+)$$

$$\left.\begin{aligned} &\text{if } E(\tilde{r}_p) \geq r_f \quad \sigma(\tilde{r}_p) = \frac{1}{\sqrt{H}}\left(E(\tilde{r}_p) - r_f\right) \\ &E(\tilde{r}_p) < r_f \quad \sigma(\tilde{r}_p) = \frac{1}{\sqrt{H}}\left(r_f - E(\tilde{r}_p)\right) \end{aligned}\right\} \text{s.e.}(\tilde{r}_p)$$

portfolio frontier in $E(\tilde{r}_p) - \sigma(\tilde{r}_p)$ space



Case 1 $r_f < \frac{A}{C}$



Consider a portfolio with expected rate of return $E(\tilde{r}_m)$
$$= \frac{A}{C} - \frac{D/C^2}{r_f - \frac{A}{C}}$$
where only risky assets are considered.

Since $r_f < \frac{A}{C}$ then $E(\tilde{r}_m) > \frac{A}{C}$.

$$\sigma(\tilde{r}_m)^2 = \frac{C}{D}\left(E(\tilde{r}_m) - \frac{A}{C}\right)^2 + \frac{1}{C} \Rightarrow \frac{dE(\tilde{r}_m)}{d\sigma(\tilde{r}_m)} = \frac{D \cdot \sigma(\tilde{r}_m)}{C\left(E(\tilde{r}_m) - \frac{A}{C}\right)} = \text{slope}$$

$$\text{slope} = \left\{\left[D^2\sigma^2(\tilde{r}_m)\right] / \left[C^2\left(E(\tilde{r}_m) - \frac{A}{C}\right)^2\right]\right\}^{\frac{1}{2}}$$

$$= \left\{\left[D^2\left(\frac{C}{D}\left(E(\tilde{r}_m) - \frac{A}{C}\right)^2 + \frac{1}{C}\right)\right] / \left[\frac{D/C}{r_f - \frac{A}{C}}\right]^2\right\}^{\frac{1}{2}}$$

$$= \left\{\left[CD\left(E(\tilde{r}_m) - \frac{A}{C}\right)^2 + \frac{D^2}{C}\right] / \left[\frac{D/C}{r_f - \frac{A}{C}}\right]^2\right\}^{\frac{1}{2}}$$

$$= \left\{\left[\frac{D}{C}\left[\frac{D/C}{r_f - \frac{A}{C}}\right]^2 + \frac{D^2}{C}\right] / \left[\frac{D/C}{r_f - \frac{A}{C}}\right]^2\right\}^{\frac{1}{2}}$$

$$= \left\{\frac{D}{C} + Cr_f^2 + \frac{A^2}{C} - 2Ar_f\right\}^{\frac{1}{2}} = \left\{Cr_f^2 - 2Ar_f + B\right\}^{\frac{1}{2}}$$

$$= \sqrt{H}$$

The intercept: $E(\tilde{r}_{zc(m)}) = r_f$

Remark

$1°$ $\theta_m^T \mathbb{1} = 1$ since θ_m is in the portfolio frontier without riskless asset

i.e. when target expected rate of return is $E(\tilde{r}_m) = \frac{A}{C} - \frac{D/C^2}{r_f - A/C}$,

the individual will invest all of his wealth on risky asset.

$2°$ when the target expected rate of return is r_f. the individual will invest nothing on risky assets.

$3°$. Black's separation:

(a). let θ_1, θ_2 be two efficient portfolios. then $\forall \alpha \in (0,1)$:

$\alpha\theta_1 + (1-\alpha)\theta_2$ is an efficient portfolio.

proof. $\theta_1 = \frac{E(\tilde{r}_1) - r_f}{H} V^{-1}(e - r_f \mathbb{1})$, $\theta_2 = \frac{E(\tilde{r}_2) - r_f}{H} V^{-1}(e - r_f \mathbb{1})$

$$\theta = \alpha\theta_1 + (1-\alpha)\theta_2 = \frac{\alpha E(\tilde{r}_1) + (1-\alpha)E(\tilde{r}_2) - r_f}{H} V^{-1}(e - r_f \mathbb{1})$$

i.e. $E(\tilde{r}) = \alpha E(\tilde{r}_1) + (1-\alpha)E(\tilde{r}_2) > r_f$ ◇

(b) let θ', θ'' be two distinct efficient portfolios, then for any efficient portfolio θ, $\exists$ unique $\alpha \in \mathbb{R}$ s.t. $\theta = \alpha\theta' + (1-\alpha)\theta'$

proof: $E(\tilde{r}_{\theta'}) \neq E(\tilde{r}_{\theta''})$ then $\exists \alpha \in \mathbb{R}$ s.t. $E(\tilde{r}_\theta) = \alpha E(\tilde{r}_{\theta'}) + (1-\alpha) E(\tilde{r}_{\theta''})$

$$\theta = \alpha\theta_1 + (1-\alpha)\theta_1 = \frac{E(\tilde{r}_\theta) - r_f}{H} V^{-1}(e - r_f \mathbb{1})$$

$$= \frac{\alpha E(\tilde{r}_1) + (1-\alpha)E(\tilde{r}_2) - r_f}{H} V^{-1}(e - r_f \mathbb{1})$$ ◇

$4°$ For any target expected rate of return $E(\tilde{r}) > r_f$. there exists an unique $\alpha > 0$ satisfying $\alpha E(\tilde{r}_m) + (1-\alpha) r_f = E(\tilde{r})$

Corresponding efficient portfolio : $\alpha\theta_m$. Proportion on riskless asset $= 1-\alpha$

Expected rate of return : $\alpha\theta_m^T e + (1-\alpha) r_f = \alpha E(\tilde{r}_m) + (1-\alpha) r_f = E(\tilde{r})$

$\sigma(\tilde{r}) = \alpha\sigma(\tilde{r}_m)$. The point $(\alpha\sigma(\tilde{r}_m), E(\tilde{r}))$ is on the efficient frontier

$$\frac{E(\tilde{r}) - r_f}{\alpha\sigma(\tilde{r}_m)} = \frac{\alpha(E(\tilde{r}_m) - r_f)}{\alpha\sigma(\tilde{r}_m)} = \frac{E(\tilde{r}_m) - r_f}{\sigma(\tilde{r}_m)} = \sqrt{H}$$

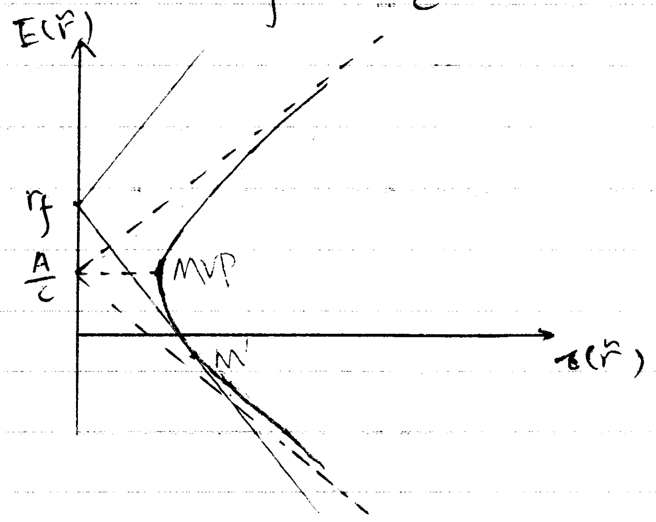
$$\begin{array}{l} \\ \\ \end{array}\begin{bmatrix} 1-\alpha \\ \cdots \\ \alpha\theta_m \end{bmatrix}\begin{array}{l} 1\times1 \\ \\ N\times1 \end{array}\quad \begin{array}{l} \text{proportion of wealth invested to riskless assets} \\ \\ \text{- - - - - - - - - - - - risky assets} \end{array}$$

5° ① Any portfolio on the line segment $\overline{r_f m}$ is a convex combination of portfolio θ_m and the riskless asset

② Any portfolio on the half line $r_f + \sqrt{H}\sigma(\tilde{r}_p)$ other than those on $\overline{r_f m}$ involves short-selling the riskless asset and investing the proceeds in portfolio θ_m

③ Any portfolio on the half line $r_f - \sqrt{H}\sigma(\tilde{r}_p)$ involves short-selling portfolio θ_m and investing the proceeds in the riskless asset.

Case 2. $r_f > \frac{A}{C}$



For any target expected rate of return $E(\tilde{r}) < r_f$. there exists a unique $\alpha > 0$

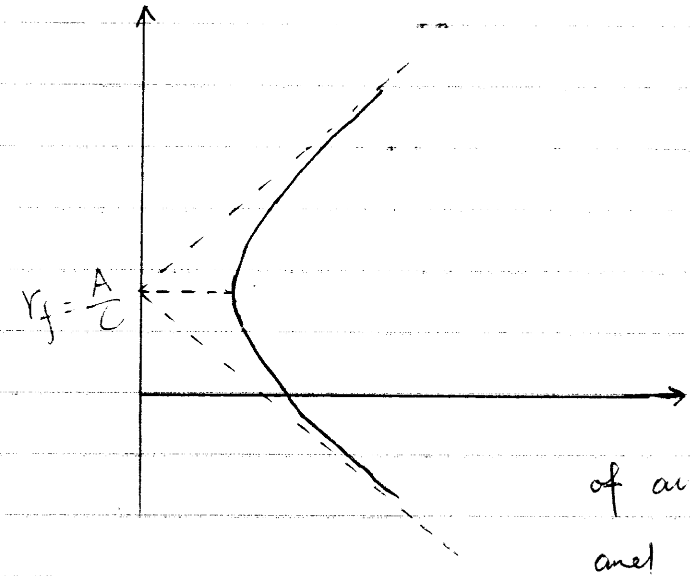
s.t. $\alpha E(\tilde{r}_{m'}) + (1-\alpha) r_f = E(\tilde{r})$

$$\alpha(E(\tilde{r}_{m'}) - r_f) = E(\tilde{r}) - r_f < 0$$

$$\begin{bmatrix} 1-\alpha \\ \text{-----} \\ \alpha\theta_{m'} \end{bmatrix}\begin{array}{l} 1\times1 \\ \\ N\times1 \end{array}\quad \begin{array}{l} \text{porportion of wealth invested to riskless assets} \\ \\ \text{- - - risky assets} \end{array}$$

Let $\beta = -\alpha$. then $\begin{bmatrix} 1+\beta \\ -\beta\theta_{m'} \end{bmatrix}$

Case 3 $r_f = \frac{A}{C}$



$$H = B - 2Ar_f + Cr_f^2$$
$$= B - 2\frac{A^2}{C} + C\cdot\frac{A^2}{C^2}$$
$$= B - \frac{A^2}{C} = \frac{D}{C}$$

No tangent portfolio: the portfolio frontier of all assets is not generated by the riskless asset and a portfolio on the portfolio frontier of risky assets.

Note: $\theta_p = V^{-1}(e - \mathbb{1} r_f) \cdot \frac{E(\tilde{r}_p) - r_f}{H}$

$$= V^{-1}(e - \mathbb{1}\cdot\frac{A}{C}) \frac{E(\tilde{r}_p) - \frac{A}{C}}{\frac{D}{C}}$$

$$\mathbb{1}^T\theta_p = \left(\mathbb{1}^T V^{-1} e - \frac{A}{C}\, \mathbb{1}^T V^{-1}\mathbb{1}\right) \frac{E(\tilde{r}_p) - \frac{A}{C}}{\frac{D}{C}}$$

$$= (A - A)\, \frac{E(\tilde{r}_p) - \frac{A}{C}}{\frac{D}{C}} = 0$$

arbitrage portfolio of risky assets: a portfolio whose weights sum to zero.

Therefore: any frontier portfolio of all assets

$$= \begin{pmatrix} 1 \\ \theta_p \end{pmatrix} \quad \text{with } \mathbb{1}^T\theta_p = 0$$

Prop The expected rate of return on any portfolio can be written as a linear combination of riskfree interest rate and expected rate of return on any frontier portfolio p.

proof: Let q be any portfolio (not necessarily on the frontier). θ_q

p be a frontier portfolio. θ_p

Assume $E(\tilde{r}_p) \neq r_f$ then

$$\mathrm{cov}(\tilde{r}_q, \tilde{r}_p) = \theta_q^T V \theta_p$$

$$= \theta_q^T V \left(V^{-1}(e - \mathbb{1} r_f) \frac{E(\tilde{r}_p) - r_f}{H} \right) = \left(\theta_q^T e - r_f \theta_q^T \mathbb{1} \right) \frac{(E(\tilde{r}_p) - r_f)}{H}$$

$$= \left(\theta_q^T e + (1 - \theta_q^T \mathbb{1}) r_f - r_f \right) \cdot \frac{E(\tilde{r}_p) - r_f}{H}$$

$$= (E(\tilde{r}_q) - r_f)(E(\tilde{r}_p) - r_f)/H$$

$$\Rightarrow E(\tilde{r}_q) - r_f = \frac{\mathrm{cov}(\tilde{r}_q, \tilde{r}_p)}{(E(\tilde{r}_p) - r_f)/H}$$

$$= \frac{\mathrm{cov}(\tilde{r}_q, \tilde{r}_p)}{(E(\tilde{r}_p) - r_f)^2/H} \cdot (E(\tilde{r}_p) - r_f)$$

$$= \underbrace{\boxed{\frac{\mathrm{cov}(\tilde{r}_q, \tilde{r}_p)}{\sigma^2(\tilde{r}_p)}}}_{\beta_{qp}} (E(\tilde{r}_p) - r_f) = \beta_{qp} [E(\tilde{r}_p) - r_f]$$

Remark: this relationship holds independent of the relation between r_f and $\frac{A}{C}$.