ECO 7377 Homework

Suggested Solution

January 21, 2026

1. To estimate β, suppose that an econometrician sets-up the following simple dummy regressor regression,

$$Y_i = \alpha_0 + \beta_0 T_i + U_1,$$

where $\alpha_0 = \mathbb{E}(Y_i(0))$ and $\beta_0 = \mathbb{E}(\beta_i)$. Write U_i as a function of $(Y_i(0), Y_i(1), T_i)$.

Solution

$$\begin{aligned} Y_i &= Y_i(0)\left(1 - T_i\right) + Y_i(1) T_i \\ &= Y_i(0) + \underbrace{\left(Y_i(1) - Y_i(0)\right)}_{=\beta_i} T_i \\ &= \alpha_0 + \beta_0 T_i + \underbrace{\left(Y_i(0) - \alpha_0\right) + \left(\beta_i - \beta_0\right) T_i}_{=U_i}. \end{aligned}$$

As a result,

$$U_i = \left(Y_i(0) - \alpha_0\right) + \left(\beta_i - \beta_0\right) T_i. \tag{1}$$

2. Is $\mathbb{E}(U_i | T_i = \tau) = 0$ for $\tau = 0, 1$?

Solution By Assumption 1.3, we have T_i is also independent of $\beta_i = Y_i(1) - Y_i(0)$ and together with (1),

$$\mathrm{E}\left(U_i | T_i = \tau\right) = \left(\alpha_0 - \alpha_0\right) + \left(\mathrm{E}\left(\beta_i\right) - \beta_0\right) T_i = 0 + 0 \times \tau = 0,$$

for both $\tau = 0, 1$.

3. Find $\mathbb{E}(U_i^2 | T_i = \tau)$ for $\tau = 0$ and $\tau = 1$.

Solution

$$\mathrm{E}\left(U_i^2 | T_i = 0\right) = \mathrm{E}\left(\left(Y_i(0) - \alpha_0\right)^2\right) = \sigma_0^2. \tag{2}$$

Note that $\beta_i - \beta_0 = (Y_i(1) - \alpha_1) - (Y_i(0) - \alpha_0)$, then

$$\begin{aligned} \mathrm{E}\left(U_i^2 | T_i = 1\right) &= \mathrm{E}\left(\left(\left(Y_i(0) - \alpha_0\right) + \left(Y_i(1) - \alpha_1\right) - \left(Y_i(0) - \alpha_0\right)\right)^2\right) \\ &= \mathrm{E}\left(\left(Y_i(1) - \alpha_1\right)^2\right) \\ &= \sigma_1^2. \end{aligned} \tag{3}$$

4. Show that the OLS estimator, $\widehat{\beta}$, of β_0 in the linear regression equation of question 1 is

$$\widehat{\beta} = \overline{Y}_1 - \overline{Y}_0,$$

where $\overline{Y}_0 = \frac{\sum_{i=1}^n Y_i(1-T_i)}{n_0}$ and $\overline{Y}_1 = \frac{\sum_{i=1}^n Y_i T_i}{n_1}$

Solution The least square estimator solves

$$\min_{\alpha,\beta} \frac{1}{2} \sum_{i=1}^n (Y_i - \alpha - \beta T_i)^2 .$$

First order conditions (we use $T_i = T_i^2$ since T_i is binary),

$$\sum_{i=1}^n T_i \left(Y_i - \hat{\alpha} - \hat{\beta} T_i\right) = 0 \Leftrightarrow n_1 \left(\bar{Y}_1 - \hat{\alpha} - \hat{\beta} = 0\right) \Leftrightarrow \hat{\alpha} = \bar{Y}_1 - \hat{\beta} \tag{4}$$

$$\sum_{i=1}^n \left(Y_i - \hat{\alpha} - \hat{\beta} T_i\right) = 0 \tag{5}$$

Plug (4) into (5), we have

$$\begin{aligned} 0 &= \sum_{i=1}^n Y_i + \left(\hat{\beta} - \bar{Y}_1\right) n - \hat{\beta} n_1 = n_0 \bar{Y}_0 + (n_1 - n)\, \bar{Y}_1 + (n - n_1)\, \hat{\beta} \\ \Rightarrow \quad & n_0 \hat{\beta} = n_0 \left(\bar{Y}_1 - \bar{Y}_0\right) \\ \Rightarrow \quad & \hat{\beta} = \bar{Y}_1 - \bar{Y}_0 \end{aligned}$$

5. Show that $\widehat{\beta}$ is consistent of β_0.

Solution

$$\begin{aligned} \bar{Y}_1 &= \frac{1}{n_1} \sum_{i=1}^n Y_i T_i = \left(\frac{1}{n} \sum_{i=1}^n T_i\right)^{-1} \left(\frac{1}{n} \sum_{i=1}^n T_i Y_i\right) \\ &= \left(\frac{1}{n} \sum_{i=1}^n T_i\right)^{-1} \left(\frac{1}{n} \sum_{i=1}^n T_i \left(\alpha_0 + \beta_0 T_i + U_i\right)\right) \\ &= \underbrace{\alpha_0 + \beta_0}_{=\alpha_0 + \alpha_1 - \alpha_0 = \alpha_1} + \left(\frac{1}{n} \sum_{i=1}^n T_i\right)^{-1} \left(\frac{1}{n} \sum_{i=1}^n T_i U_i\right) \\ &= \alpha_1 + \left(\frac{1}{n} \sum_{i=1}^n T_i\right)^{-1} \left(\frac{1}{n} \sum_{i=1}^n T_i U_i\right) \end{aligned} \tag{6}$$

By law of iterated expectations and results in question 2,

$$\mathrm{E}\left(T_i U_i\right) = \mathrm{E}\left(\mathrm{E}\left(U_i | T_i\right) T_i\right) = 0.$$

We also have $\mathrm{E}\left(T_i\right) = p$ by Assumption 2. By law of large number (LLN), we have $n^{-1} \sum_{i=1}^n T_i \to_p p$, $n^{-1} \sum_{i=1}^n T_i U_i \to 0$. By continuous mapping theorem (CMT) and Slutsky theorem, we have,

$$\bar{Y}_1 = \alpha_1 + \underbrace{\left(\frac{1}{n} \sum_{i=1}^n T_i\right)^{-1}}_{\to_p\, p^{-1}} \underbrace{\left(\frac{1}{n} \sum_{i=1}^n T_i U_i\right)}_{\to_p 0} \to_p \alpha_1. \tag{7}$$

Denote $C_i = 1 - T_i$, which implies $C_i T_i = 0$.

$$\begin{aligned}
\bar{Y}_0 &= \frac{1}{n_0}\sum_{i=1}^{n} Y_i C_i = \left(\frac{1}{n}\sum_{i=1}^{n} C_i\right)^{-1}\left(\frac{1}{n}\sum_{i=1}^{n} C_i Y_i\right) \\
&= \left(\frac{1}{n}\sum_{i=1}^{n} C_i\right)^{-1}\left(\frac{1}{n}\sum_{i=1}^{n} C_i\left(\alpha_0 + \beta_0 T_i + U_i\right)\right) \\
&= \alpha_0 + \underbrace{\left(\frac{1}{n}\sum_{i=1}^{n} C_i\right)^{-1}\left(\frac{1}{n}\sum_{i=1}^{n} C_i T_i\right)\beta_0}_{=0} + \left(\frac{1}{n}\sum_{i=1}^{n} C_i\right)^{-1}\left(\frac{1}{n}\sum_{i=1}^{n} C_i U_i\right) \\
&= \alpha_1 + \left(\frac{1}{n}\sum_{i=1}^{n} C_i\right)^{-1}\left(\frac{1}{n}\sum_{i=1}^{n} C_i U_i\right). \qquad (8)
\end{aligned}$$

Similar as steps for $\bar{Y}_1$, we have

$$\mathrm{E}\left(C_i U_i\right) = \mathrm{E}\left(\mathrm{E}\left(U_i | C_i\right)\right) = 0.$$

and $\mathrm{E}\left(C_i\right) = 1 - p$, then

$$\bar{Y}_0 = \alpha_0 + \underbrace{\left(\frac{1}{n}\sum_{i=1}^{n} C_i\right)^{-1}}_{\to_p (1-p)^{-1}}\underbrace{\left(\frac{1}{n}\sum_{i=1}^{n} C_i U_i\right)}_{\to_p 0} \to_p \alpha_0. \qquad (9)$$

By Slutsky theorem, we have

$$\hat{\beta} = \bar{Y}_1 - \bar{Y}_0 \to_p \alpha_1 - \alpha_0$$

as $n \to \infty$.

6. Derive the asymptotic distribution of $\widehat{\beta}$.

By (6) and (8), we have

$$\begin{aligned}
\sqrt{n}\left(\hat{\beta} - \beta_0\right) &= \sqrt{n}\left(\bar{Y}_1 - \alpha_1\right) - \sqrt{n}\left(\bar{Y}_0 - \alpha_0\right) \\
&= \left(\frac{1}{n}\sum_{i=1}^{n} T_i\right)^{-1}\left(\frac{1}{\sqrt{n}}\sum_{i=1}^{n} T_i U_i\right) - \left(\frac{1}{n}\sum_{i=1}^{n} C_i\right)^{-1}\left(\frac{1}{\sqrt{n}}\sum_{i=1}^{n} C_i U_i\right) \\
&= p^{-1}\left(\frac{1}{\sqrt{n}}\sum_{i=1}^{n} T_i U_i\right) - (1-p)^{-1}\left(\frac{1}{\sqrt{n}}\sum_{i=1}^{n} C_i U_i\right) \\
&\quad + \underbrace{\left(\left(\frac{1}{n}\sum_{i=1}^{n} T_i\right)^{-1} - p^{-1}\right)\left(\frac{1}{\sqrt{n}}\sum_{i=1}^{n} T_i U_i\right)}_{=I} \\
&\quad + \underbrace{\left(\left(\frac{1}{n}\sum_{i=1}^{n} C_i\right)^{-1} - (1-p)^{-1}\right)\left(\frac{1}{\sqrt{n}}\sum_{i=1}^{n} C_i U_i\right)}_{=II} \\
&= \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left(p^{-1} T_i U_i - (1-p)^{-1} C_i U_i\right) + I + II
\end{aligned}$$

By (2) and (3), law of iterated expectations, and $\mathrm{E}(T_iU_i) = 0$, we have

$$\mathrm{Var}(T_iU_i) = \mathrm{E}\left((T_iU_i)^2\right) = \mathrm{E}\left(\mathrm{E}\left(U_i^2|T_i\right)T_i\right) = p\sigma_1^2.$$

Similarly, we have $\mathrm{Var}(C_iU_i) = (1-p)\sigma_0^2$. Then by LLN as in question 5 and central limit theorem (CLT), we have,

$$I = \underbrace{\left(\left(\frac{1}{n}\sum_{i=1}^{n} T_i\right)^{-1} - p^{-1}\right)}_{\to_p 0} \underbrace{\left(\frac{1}{\sqrt{n}}\sum_{i=1}^{n} T_iU_i\right)}_{\to_d N(0,p\sigma_1^2)} = o_p(1)O_p(1) = o_p(1).$$

Similarly, $II = o_p(1)$. Denote $v_i = p^{-1}T_iU_i - (1-p)^{-1}C_iU_i$, then we have $\mathrm{E}(v_i) = 0$ and

$$\mathrm{Var}(v_i) = \mathrm{E}\left(\mathrm{E}\left(v_i^2|T_i\right)\right) = \mathrm{E}\left(\left(p^{-2}T_i + (1-p)^{-2}(1-T_i)\right)\mathrm{E}\left(U_i^2|T_i\right)\right) = p^{-1}\sigma_1^2 + (1-p)^{-1}\sigma_0^2.$$

Then by CLT and generalized Slutsky theorem, we have

$$\sqrt{n}\left(\hat{\beta} - \beta_0\right) \to_d N\left(0, p^{-1}\sigma_1^2 + (1-p)^{-1}\sigma_0^2\right),$$

as $n \to \infty$.

7. Provide a consistent estimator of the limit variance you derive in question 6.

Solution Let $\hat{p} = n_1/n$, $\hat{\sigma}_1^2 = n_1^{-1}\sum_{i=1}^{n}\left(Y_i - \bar{Y}_1\right)^2 T_i$, $\hat{\sigma}_0^2 = n_0^{-1}\sum_{i=1}^{n}\left(Y_i - \bar{Y}_0\right)^2(1-T_i)$. We propose to use $\widehat{V} = \hat{p}^{-1}\hat{\sigma}_1^2 + (1-\hat{p})^{-1}\hat{\sigma}_0^2$ as the estimator of the limit variance.

8. Suppose that $p = 1/2$. Suppose that the econometrician constructs the two-sided 95% confidence interval using the standard error assuming homoskedasticity. What is the asymptotic coverage probabilily of the CI? Is it valid?

Solution

Suppose the econometrician uses the standard error assuming homoskedasticity, the variance estimator is

$$\begin{aligned}
\hat{V}_{homo} &= \left[\left(\frac{1}{n}\sum_{i=1}^{n}(1,T_i)'(1,T_i)\right)^{-1}\right]_{2,2}\hat{\sigma}^2 \\
&= \left[\begin{pmatrix} 1 & \frac{1}{n}\sum_{i=1}^{n} T_i \\ \frac{1}{n}\sum_{i=1}^{n} T_i & \frac{1}{n}\sum_{i=1}^{n} T_i \end{pmatrix}^{-1}\right]_{2,2}\hat{\sigma}^2
\end{aligned}$$

where

$$\begin{aligned}
\hat{\sigma}^2 &= \frac{1}{n}\sum_{i=1}^{n}\left(Y_i - \bar{Y}_0 - \left(\bar{Y}_1 - \bar{Y}_0\right)T_i\right)^2 \\
&= \frac{1}{n}\sum_{i=1}^{n}\left(\left(Y_i - \bar{Y}_0\right)(1-T_i)\right)^2 + \frac{1}{n}\sum_{i=1}^{n}\left(\left(Y_i - \bar{Y}_1\right)T_i\right)^2 \\
&\to_p p\sigma_1^2 + (1-p)\sigma_0^2,
\end{aligned}$$

$$\left[\begin{pmatrix} 1 & \frac{1}{n}\sum_{i=1}^{n} T_i \\ \frac{1}{n}\sum_{i=1}^{n} T_i & \frac{1}{n}\sum_{i=1}^{n} T_i \end{pmatrix}^{-1}\right]_{2,2} \to_p \left[\begin{pmatrix} 1 & p \\ p & p \end{pmatrix}^{-1}\right]_{2,2} = p(1-p),$$

as $n \to \infty$. As a result, $\hat{V}_{homo} \to_p (1-p)^{-1}\sigma_1^2 + p^{-1}\sigma_0^2$.

When $p = 1/2$, the asymptotic variance is the same as in the correct heteroskedasticity case in problem 6, so the asymptotic coverage probability should be 95%.

Instead of the random assignment of T_i as in Assumption 3, the treatment is assigned as $T_i = 1$ if $Y_i(1) > Y_i(0)$ and $T_i = 0$ otherwise.

9. Is $\mathbb{E}(U_i|T_i) = 0$?

Solution Note that

$$\mathrm{E}\left(U_i|T_i = \tau\right) = \left(\alpha_0 - \alpha_0\right) + \left(\mathrm{E}\left(\beta_i|T_i\right) - \beta_0\right) T_i \begin{cases} = 0 & \tau = 0 \\ > 0 & \tau = 1 \end{cases}$$

So $\mathbb{E}(U_i|T_i) \neq 0$.

10. Is $\widehat{\beta}$ is consistent of β_0?

Solution No longer consistent since $\frac{1}{n}\sum_{i=1}^{n} T_i U_i$ does not converge to 0 in probabilily as in problem 5.